Technical Report 364

K. J. Harte

Spin-Wave Effects in the Magnetization Reversal of a Thin Ferromagnetic Film

27 August 1964

Prepared under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the support of the U.S. Air Force under Contract AF 19(628)-500.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

SPIN-WAVE EFFECTS IN THE MAGNETIZATION REVERSAL OF A THIN FERROMAGNETIC FILM

K. J. HARTE

Group 24

TECHNICAL REPORT 364

27 AUGUST 1964

LEXINGTON MASSACHUSETTS

SPIN-WAVE EFFECTS IN THE MAGNETIZATION REVERSAL OF A THIN FERROMAGNETIC FILM*

ABSTRACT

The influence of spin waves on rapid rotational magnetization reversal (switching) in a thin ferromagnetic film is investigated by means of a semiclassical, continuum theory which includes external, anisotropy, exchange, and magnetostatic (dipolar) fields. To simplify the magnetostatic field, a "thin-film approximation" is introduced, in which the magnetization is replaced by its average over the film thickness. From a stochastic model for the microstructure of a polycrystalline film, the equilibrium magnetization configuration $\vec{M}(\vec{r})$ is derived. Planar fluctuations of $\vec{M}$ from its mean direction $\vec{m}_o$ are found which have the characteristics of "longitudinal ripple," namely, wave vectors $\vec{k}$ parallel to $\vec{m}_o$ and wavelengths greater than an exchange cutoff $2\pi\lambda_e \sim 10^{-4}$ cm. Components with wave vectors in directions other than $\pm\vec{m}_o$ are attenuated by magnetostatic forces, while exchange forces attenuate components with wavelengths less than $2\pi\lambda_e$. The magnetization dispersion δ [rms angular deviation of $\vec{M}(\vec{r})$ from $\vec{m}_o$] is also calculated. A brief discussion is given of the uniform rotational switching mode (without spin waves), with particular attention to undamped and overdamped cases. From the spin-wave equations of motion, the spectrum applicable to a parallel resonance situation (external field in the film plane) is first obtained, and long-wavelength magnetostatic distortion is noted. Then the transient spin-wave response is calculated for a switching field $\vec{H}_p$ suddenly turned on at $t = 0$. It is found that if $\vec{m}_o(t)$ rotates faster than longitudinal spin waves [$\vec{k} \parallel \vec{m}_o(0)$] can relax, the magnetization goes through a transient state of high magnetostatic energy, and a spin-wave reaction torque (proportional to δ^2) is exerted on the uniform mode. If H_p is less than a critical field H_{pc}, the reaction torque at some point in the switching process is greater than the reversing torque and the uniform mode becomes locked; rotational reversal cannot proceed until initially longitudinal spin waves have relaxed into components propagating in the instantaneous direction of $\vec{m}_o(t)$. Such a highly damped process is suggestive of the noncoherent reversal mode observed in thin films. For $H_p > H_{pc}$, reversal takes place by a modified uniform rotation; H_{pc} may therefore be identified as the threshold field for coherent rotation. The calculated dependence of H_{pc} on a bias field compares favorably with experiment. The δ-dependence, if δ can be measured independently, should provide a crucial test of the theory.

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
Chief, Lincoln Laboratory Office

---

*This report is based on a thesis of the same title submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the Department of Physics at Harvard University.

TABLE OF CONTENTS

	Abstract	iii
I.	Introduction	1
	A. Magnetization Reversal and Thin Films: Experimental Background	1
	B. Theoretical Foundation	4
II.	Magnetostatic Field of a Nonuniformly Magnetized Thin Film	7
III.	Static Equilibrium: Magnetization Ripple	13
	A. Introduction: Experimental Background	13
	B. Random Anisotropy Model	14
	C. Magnetization Ripple Spectrum	19
	D. Discussion of Result	23
	E. Magnetization Dispersion	27
IV.	Uniform Magnetization Reversal	31
	A. Introduction	31
	B. Equations of Motion	31
	C. Undamped Uniform Mode	37
	D. Overdamped Uniform Mode	38
V.	Spin-Wave Theory: Quasistatic Spectrum and Transient Response	43
	A. Introduction	43
	B. Equations of Motion	44
	C. Quasistatic Spectrum	48
	D. Transient Response	52
VI.	Effect of Spin Waves on Magnetization Reversal	61
	A. Introduction	61
	B. Nonlinear Reaction Torque	61
	C. Rotational Magnetization Reversal: Dynamic Locking	66
Appendix A – Notation; Parameters of a TPF		71
	I. Notation	71
	II. Parameters of a Typical Permalloy Film (TPF)	71
Appendix B – Coupling of Ripple Components		73
Appendix C – On the WKB Method		77
Appendix D – The Critical Field		81
	I. Hard-Axis Bias Field – Easy-Axis Pulse Field	81
	II. Zero Bias Field	83

SPIN-WAVE EFFECTS IN THE MAGNETIZATION REVERSAL OF A THIN FERROMAGNETIC FILM

I. INTRODUCTION

A. MAGNETIZATION REVERSAL AND THIN FILMS: EXPERIMENTAL BACKGROUND

In the past decade considerable interest has arisen in the phenomenon of magnetization reversal in ferromagnetic media, i.e., an irreversible change in the direction of magnetization following the application of a pulse magnetic field. Although this interest received its initial impetus from the purely practical possibility of ferromagnetic storage elements for high-speed computers, it soon became apparent that the physical processes involved were either imperfectly understood or else completely unknown. The basic distinction between magnetization reversal, or switching, on the one hand, and various magnetic resonance phenomena, which are generally much better understood, on the other, is that the former involves large displacements of the magnetization $\vec{M}$ from its equilibrium direction. How $\vec{M}$ returns to equilibrium is the problem to which we shall devote ourselves in this work.

In particular, we shall be concerned with magnetization reversal in thin films, for the resultant theoretical simplicity seems well worth the loss in generality. Furthermore, experimental results with thin films tend to be consistent and unambiguous, in contrast to the somewhat confused situation with respect to switching in bulk ferromagnets. Among the significant properties of thin films are: (a) the shape demagnetizing field confines $\vec{M}$ to the film plane or nearly so (in the absence of a strong field or easy axis normal to the plane); (b) eddy-current effects in ferromagnetic metals may be eliminated by choosing a film thickness much less than the skin depth; (c) the equilibrium magnetic configuration may be directly observed since it is invariably the same as the surface configuration; (d) the typically small crystallite size ($\sim$300 Å) of polycrystalline films, in conjunction with (a), results in a remanent state which is very nearly single-domain; (e) single-crystal films of various ferromagnetic metals and nonmetals may be studied (these are usually epitaxially grown); (f) low anisotropy (a few oersteds) is readily obtained, thus allowing magnetization reversal to take place in low, fast-rise fields.

Since the early experiments of Blois,[1] many workers have studied pulse switching in thin films,[2-10] usually of Permalloy (Ni-Fe alloy near the zero magnetostriction 0.83-0.17 composition), and possessing uniaxial anisotropy with easy axis in the film plane. In very idealized form, these experiments have been performed in the following manner (see the above references for details, especially Refs. 5 and 9). A fast rise-time ($\lesssim 10^{-9}$ sec) pulse field $\vec{H}_p$ of a few oersteds magnitude is applied in the film plane at an obtuse angle to the mean magnetization $\vec{m}_o$ of an initially single-domain film.† The switching time t_s and other information on the reversal

†Many of the early experiments were performed with $\vec{H}_p$ antiparallel to $\vec{m}_o$, resulting often in bi-directional reversal, with its own peculiar characteristics. We shall exclude this limiting case from our treatment.

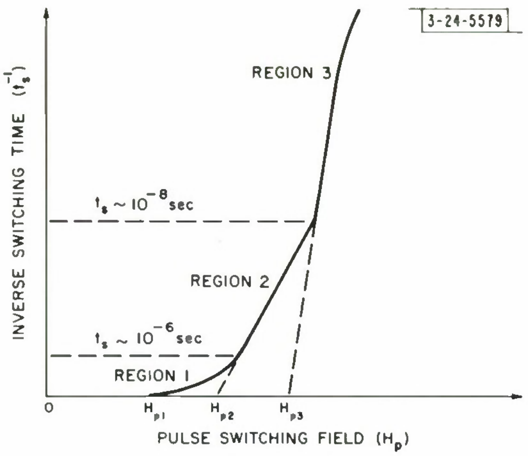

Fig. 1. Schemotic illustrotion of o thin-film switching curve showing (1) low-speed region, (2) intermediote-speed region, ond (3) high-speed region. Also shown ore threshold fields for eoch region.

process is obtained from the voltage induced in a pickup loop with axis parallel to $\vec{H}_p$. This voltage is termed the longitudinal switching signal; further information may be obtained from the transverse switching signal, induced in an orthogonal pickup loop with axis also in the film plane.†

The main results of these experiments are conveniently displayed in the form of switching curves, in which inverse switching time is plotted as a function of the pulse field. Figure 1 shows a typical thin-film switching curve; we note the presence of three distinct regions (first observed by Olson and Pohm[5]). The reversal process in region 1 (low speed: $t_s \gtrsim 10^{-6}$ sec) has been identified as the nucleation of reverse domains and subsequent domain-wall motion. The threshold field for this process, H_{p1}, is the wall coercive force; a fairly successful model for domain-wall switching has been proposed by Conger and Essig.[11] We shall not discuss region 1 further since the physical processes involved are well understood.

We proceed now to region 3 (high speed: $t_s \lesssim 10^{-8}$ sec), where the reversal process is clearly a coherent, or nearly coherent, rotation of $\vec{M}$. A phenomenological theory of such a process, based on a damped gyromagnetic equation such as that proposed by Landau and Lifshitz,[12] has been found by Smith,[4] and by Olson and Pohm,[5] to describe switching in region 3 fairly well. However, there are two difficulties. First, and most disturbing, the threshold field H_{p3} is always found to be greater (by a factor ~1.5) than the theoretical Stoner-Wohlfarth[13] threshold H_{pt} for a uniformly magnetized uniaxial film. Second, the damping of high-speed reversal is found to be 3 or 4 times the intrinsic damping, as obtained from low-power ferromagnetic resonance linewidth[10,14,15] or decay of free oscillations of the magnetization.[10,14]

Smith and Harte[15] have suggested that the discrepancy in the threshold field may be caused by dispersion in the direction and amplitude of the uniaxial anisotropy.[16,17] By assuming that those regions of a film with the highest threshold control coherent rotation, they have been able to account for observed thresholds with fairly plausible values of anisotropy dispersion. However, their assumption is difficult to justify, since the effective anisotropy in coherent rotation of $\vec{M}$ should be the mean, or close to it. Furthermore, interactions between regions were neglected. The discrepancy in damping, for which no explanation exists, implies that there is a loss mechanism operative in (large-angle) switching that is not present in (small-angle) resonance.

† The exoct meoning of "switching time" vories somewhot from worker to worker ond moy depend upon the noture of the reversol process. For present purposes, t_s need not be precisely defined and moy just be considered os some meosure of the durotion of the longitudinol switching signol.

It should now be clear that even such a conceptually simple process as coherent domain rotation is only imperfectly understood.

Region 2 (intermediate speed: $10^{-8} \lesssim t_s \lesssim 10^{-6}$) has baffled workers since its discovery[2,5] and the physical processes involved are not known. The threshold field H_{p2}, however, has been found to coincide with the theoretical rotational threshold H_{pt}.[4,5] This, in addition to the observation of a sizable transverse switching signal, implies that reversal takes place at least partially by a rotational mechanism. However, switching in region 2 is about twenty times slower than that predicted by the above-mentioned theory of coherent reversal. Humphrey[6] has suggested that reversal starts by simple coherent rotation but quickly breaks up into what he has termed noncoherent rotation. Although this picture is consistent with all experimental evidence, including the shape of the longitudinal switching signal (an initial spike followed by a long tail[5]), a physical model was lacking, and it was not even clear what precipitated the breakup.

Humphrey and Gyorgy[8] have obtained a phenomenological description of noncoherent rotation based on the Gilbert[18] modification of the Landau-Lifshitz equation with damping about 100 times the intrinsic damping, but no explanation of this extraordinarily large loss was given. Harte[19] has proposed a model of noncoherent rotation based on angular anisotropy dispersion, but magnetostatic interactions were neglected and the remaining effects are far too small to account for the large loss. Smith and Harte[15] have suggested that intermediate-speed switching may take place by a sequential process, observed quasistatically by Smith,[20] which they termed labyrinth propagation. (An incipient labyrinth may be seen in the photograph in Fig. 2 as a reverse domain which has propagated from the lower right film edge.) Another possibility is a partial rotation process, observed quasistatically by Methfessel, et al.[21] However, no attempt was made in either case to calculate the dynamic characteristics of such processes, and it is not known how valid these models are.

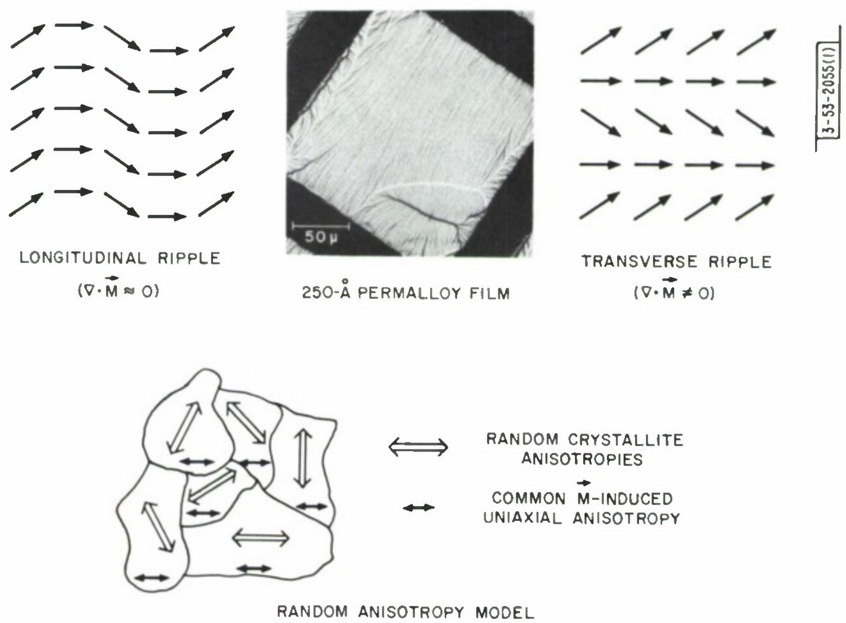

Fig. 2. Magnetization ripple in a thin film. Top center: defocused electron micrograph of a 250-Å Permalloy film (courtesy of M. S. Cohen) showing longitudinal ripple (fine structure) and large-angle domain walls (heavy black or white lines). Top left and right: schematic illustration of longitudinal and transverse ripple, respectively. Bottom: anisotropy model showing randomly varying and uniform uniaxial anisotropies.

In summary, the unsolved problems in thin-film switching are: (a) accounting for the elevated coherent rotational threshold; (b) discovering the reason for breakup of coherent rotation in region 2; (c) finding the origin of the anomalous damping in coherent rotation; and (d) uncovering the physical mechanism of noncoherent rotation. The theory we develop provides answers to (a) and (b), provides a qualitative answer to (c) – further calculation should result in a quantitative comparison of theory and experiment on this point, and leads to a suggestion for (d) which, although plausible, cannot be fairly judged without additional development of the model.

B. THEORETICAL FOUNDATION

Our analysis of magnetization reversal is based on a semiclassical, continuum theory in which the quantum mechanical spin-density operator of the ferromagnetic electrons is replaced by a constant times a classical vector field $\vec{M}(\vec{r}, t)$ – the magnetic moment density or magnetization. This replacement is possible provided the wavelengths of all significant disturbances of the spin system are much greater than the lattice constant and provided we are not too near the Curie temperature. For a thorough discussion of the validity of this semiclassical theory, the reader is referred to a paper by Herring and Kittel.[22] We assume that all fluctuations of $\vec{M}(\vec{r}, t)$ from its spatial average $\vec{m}_o(t)$ are small, i.e., that the sample is nearly a single domain. This condition we write as

$$|\delta\vec{M}| << M_o \tag{I-1a}$$

where

$$\vec{M}(\vec{r}, t) = \vec{m}_o(t) + \delta\vec{M}(\vec{r}, t) \tag{I-1b}$$

and

$$M_o = |\vec{M}| \quad . \tag{I-1c}$$

The fluctuations $\delta\vec{M}$ will be assumed constant across the film thickness, i.e., with wave vectors $\vec{k}$ in the film plane. However, generalization to three-dimensional $\vec{k}$-space is not difficult so that it may be possible to generalize the theory to include reversal processes in bulk ferromagnets.

As suggested phenomenologically by Landau and Lifshitz,[12] and later established on a very general basis by Herring and Kittel,[22] the equation of motion for $\vec{M}$ may be written in the form

$$\frac{\partial\vec{M}}{\partial t} = -\gamma\vec{M} \times \vec{H}_{eff} + \text{damping term} \tag{I-2}$$

where γ is the absolute value of the gyromagnetic ratio and $\vec{H}_{eff}(\vec{r}, t)$ is an effective magnetic field which, for our purposes, consists of an exchange field $\vec{H}_e$, a magnetostatic field $\vec{H}_m$, an effective anisotropy field $\vec{H}_a$, and an external field $\vec{H}_h$. In addition, $\vec{H}_{eff}$ includes the effects of magnetostriction and finite conductivity. In the theory we shall develop, damping processes are of secondary importance; therefore the form of the damping term in (I-2) is not crucial to our work. In Sec. IV we shall briefly discuss the various damping terms which have been proposed, but for most of this work damping will be neglected.

The constant γ is given by

$$\gamma = -\frac{ge}{2mc} \tag{I-3}$$

where e and m are the electronic charge and mass, c is the velocity of light, and the spectroscopic splitting factor g is slightly greater than 2 due to incomplete quenching of orbital angular momentum.[23]

The exchange field $\vec{H}_e$, which arises from the short-range interactions responsible for ferromagnetism, has been shown by Herring and Kittel[22] to be given, for a cubic lattice, by

$$\vec{H}_e = \frac{2A}{M_o^2} \nabla^2 \vec{M} \tag{I-4}$$

where A is an exchange constant (in erg/cm). It should be noted that $\vec{H}_e$, as given by (I-4), is independent of atomic model and follows from the energy expression of lowest order in the spatial derivatives of $\vec{M}$ allowed by the symmetry of the lattice, and invariant with respect to spin rotations and inversions.

The magnetostatic field $\vec{H}_m$ has its origin in dipole-dipole interactions and in our continuum theory may be found from Maxwell's magnetostatic equations

$$\nabla \cdot (\vec{H}_m + 4\pi\vec{M}) = 0 \tag{I-5a}$$

$$\nabla \times \vec{H}_m = 0 \quad . \tag{I-5b}$$

We devote Sec. II to the solution of (I-5a, b) with boundary conditions appropriate to thin-film geometry.

Anisotropy forces in thin polycrystalline films are of two general types, uniform and local. A uniform uniaxial anisotropy is usually found to be induced along the magnetization direction during film deposition.[1] Although the physical origins of this anisotropy are not completely understood (for a discussion of these the reader is referred to a paper by Smith[20]), a satisfactory phenomenological description may be obtained from the energy expression

$$E_a = -K_o M_o^{-2} (\vec{M} \cdot \vec{i}_x)^2 \tag{I-6}$$

where $\vec{i}_x$ is a unit vector along the easy axis and K_o is a positive constant. Local anisotropy forces arise from inhomogeneities and will be discussed in the introduction to Sec. III. Magnetostrictive effects, whether uniform or local, will be included in the effective anisotropy field.

External fields may be steady or time-varying and entail no conceptual problems. Rise times of the order of 10^{-10} sec for a pulse field of a few oersteds are now readily available, and we shall assume that pulse switching fields may be represented by step functions.

The primary effect of finite conductivity in metallic films is eddy-current damping. Smith[24] has shown that the mean eddy-current contribution to the effective damping α [see (IV-3)] is given by

$$\alpha_e = \frac{4\pi L^2 \omega_m}{3\rho} \tag{I-7}$$

where L is the half-thickness of the film, ρ is the resistivity, and $\omega_m = 4\pi M_o \gamma$. For a Permalloy film with $\rho = 2 \times 10^4$ abohm cm, α_e is equal to the intrinsic damping of about 10^{-2} for $L = 1.6 \times 10^{-5}$ cm. Thus, unless the film thickness is greater than about 2500 Å, eddy currents may be neglected. Where necessary, α will be assumed to include α_e, but we shall not be concerned explicitly with this effect in the theory to follow.

We remark here that we shall find it useful to introduce the concept of a "typical Permalloy film" (TPF), the properties of which are listed in Appendix A. This we do mainly for the purpose of numerical estimates of our results, but a few of the approximations we use would have to be altered in order to apply our theory to a film with parameters drastically different from a TPF.

II. MAGNETOSTATIC FIELD OF A NONUNIFORMLY MAGNETIZED THIN FILM

A fundamental source of difficulty in any magnetic problem involving fluctuations of the magnetization $\vec{M}(\vec{r})$ on a scale comparable to a sample dimension is the magnetostatic (dipole-dipole) interaction. This long range force leads to a field $\vec{H}(\vec{r})$ which depends not only on $\vec{M}(\vec{r})$ but also on $\vec{M}$ throughout the sample.† In conjunction with other forces such as exchange and anisotropy, it usually renders static calculations very difficult (only a few exactly soluble cases are known[24]) and dynamic calculations intractable. Thin-film geometry, however, greatly simplifies the magnetostatic problem, enabling us to find an approximate solution for $\vec{H}(\vec{r})$ in a form which is very convenient for both static and dynamic situations.

We calculate this field (also called the stray, dipolar, or demagnetizing field) from Maxwell's magnetostatic equations, for a planar film of thickness 2L with an arbitrary magnetization distribution $\vec{M}(\vec{r})$. A rectangular coordinate system is used with film boundaries given by $x = \pm L_x$, $y = \pm L_y$, and $z = \pm L$ (z-axis normal to film plane). We eventually go to the limit $L_{x,y}L^{-1} \to \infty$, so that edge effects may be ignored, i.e., the field from poles on the film rim is neglected. For convenience in this section, we introduce a normalized position vector $\vec{\sigma} = \vec{r}/L = (\xi, \eta, \zeta)$, and use the notations $\nabla \equiv (\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\zeta)$ and $f'(\vec{\sigma}) \equiv \partial f/\partial\zeta$. Maxwell's magnetostatic equations are given by (I-5a, b):

$$\nabla \cdot \vec{B}(\vec{\sigma}) = 0 \tag{II-1a}$$

$$\nabla \times \vec{H}(\vec{\sigma}) = 0 \tag{II-1b}$$

where

$$\vec{B} \equiv \vec{H} + 4\pi\vec{M} \quad . \tag{II-1c}$$

From (II-1a) and (II-1b) we have the boundary conditions

$$\vec{B} \cdot \vec{i}_z \text{ continuous at } \zeta = \pm 1 \tag{II-2a}$$

$$\vec{H} \times \vec{i}_z \text{ continuous at } \zeta = \pm 1 \tag{II-2b}$$

where $\vec{i}_z$ is a unit vector in the z-direction. Also, we require

$$\lim_{|\zeta|\to\infty} \vec{H}(\vec{\sigma}) = 0 \quad . \tag{II-3}$$

We write $\vec{M}$ in the form

$$\begin{aligned} \vec{M}(\vec{\sigma}) &= \sum_{\vec{\kappa}} \vec{m}_{\vec{\kappa}}(\zeta)\, e^{i\vec{\kappa}\cdot\vec{\sigma}} && -1 \leqslant \zeta \leqslant 1 \\ &= 0 && |\zeta| > 1 \end{aligned} \tag{II-4}$$

† In this section we omit the subscript on $\vec{H}_m$ for brevity.

where

$$\vec{\kappa} = \vec{k}L = \pi L\left(\frac{n_x}{L_x}\,,\ \frac{n_y}{L_y}\,,\ 0\right)\ (n_x, n_y = 0, \pm 1, \pm 2, \ldots) \quad . \tag{II-5}$$

From (II-1b), $\vec{H} = -\nabla\phi$, so that the potential $\phi(\vec{\sigma})$ satisfies

$$\nabla^2\phi(\vec{\sigma}) = 4\pi\nabla\cdot\vec{M} = 4\pi\sum_{\vec{\kappa}}[\vec{i}_z\cdot\vec{m}'_{\vec{\kappa}}(\zeta) + i\vec{\kappa}\cdot\vec{m}_{\vec{\kappa}}(\zeta)]\,e^{i\vec{\kappa}\cdot\vec{\sigma}} \qquad -1\leqslant\zeta\leqslant 1$$

$$= 0 \qquad |\zeta| > 1 \quad . \tag{II-6}$$

The problem now consists of solving (II-6) with the boundary conditions

$$(4\pi\,\vec{i}_z\cdot\vec{M} - \phi')\Big|_{\zeta=\pm 1} \quad \text{continuous} \tag{II-7a}$$

$$\frac{\partial\phi}{\partial x}\Big|_{\zeta=\pm 1},\ \frac{\partial\phi}{\partial y}\Big|_{\zeta=\pm 1} \quad \text{continuous} \tag{II-7b}$$

and

$$\lim_{|\zeta|\to\infty}\phi = \text{const.} \tag{II-7c}$$

We adopt a description of the potential ϕ of the form (II-4):

$$\phi(\vec{\sigma}) = \sum_{\vec{\kappa}}\phi_{\vec{\kappa}}(\zeta)\,e^{i\vec{\kappa}\cdot\vec{\sigma}} \qquad -1\leqslant\zeta\leqslant 1$$

$$= \sum_{\vec{\kappa}}\phi^{\pm}_{\vec{\kappa}}(\zeta)\,e^{i\vec{\kappa}\cdot\vec{\sigma}} \qquad \begin{matrix}\phi^{+}\ \text{for}\ \zeta > 1\\ \phi^{-}\ \text{for}\ \zeta < -1\end{matrix} \quad . \tag{II-8}$$

Then from (II-6) and (II-8)

$$-\kappa^2\phi_{\vec{\kappa}}(\zeta) + \phi''_{\vec{\kappa}}(\zeta) = 4\pi\rho_{\vec{\kappa}}(\zeta) \tag{II-9a}$$

$$-\kappa^2\phi^{\pm}_{\vec{\kappa}}(\zeta) + \phi^{\pm''}_{\vec{\kappa}}(\zeta) = 0 \tag{II-9b}$$

where $\kappa = |\vec{\kappa}|$ and

$$\rho_{\vec{\kappa}}(\zeta) = \vec{i}_z\cdot\vec{m}'_{\vec{\kappa}}(\zeta) + i\vec{\kappa}\cdot\vec{m}_{\vec{\kappa}}(\zeta) \tag{II-10}$$

(the volume pole density). The transformed boundary conditions are

$$4\pi\,\vec{i}_z\cdot\vec{m}_{\vec{\kappa}}(\pm 1) - \phi'_{\vec{\kappa}}(\pm 1) = -\phi^{\pm'}_{\vec{\kappa}}(\pm 1) \tag{II-11a}$$

$$\phi_{\vec{\kappa}}(\pm 1) = \phi^{\pm}_{\vec{\kappa}}(\pm 1) \tag{II-11b}$$

$$\lim_{\zeta\to\pm\infty}\phi^{\pm}_{\vec{\kappa}}(\zeta) = \text{const.} \tag{II-11c}$$

For $\vec{\kappa} = 0$, the solutions are easily found to be

$$\phi_o = 4\pi \vec{i}_z \cdot \int_0^{\zeta} \vec{m}_o(\nu)\, d\nu \qquad \text{(II-12b)}$$

$$\phi_o^+ = 4\pi \vec{i}_z \cdot \int_0^{1} \vec{m}_o(\zeta)\, d\zeta \qquad \text{(II-12b)}$$

$$\phi_o^- = 4\pi \vec{i}_z \cdot \int_0^{-1} \vec{m}_o(\zeta)\, d\zeta \qquad \text{(II-12c)}$$

to within an arbitrary additive constant. For $\vec{\kappa} \neq 0$, the solution to (II-9a) is

$$\phi_{\vec{\kappa}} = A_{\vec{\kappa}}\, e^{\kappa\zeta} + B_{\vec{\kappa}}\, e^{-\kappa\zeta} + \frac{4\pi}{\kappa} \int_0^{\zeta} \rho_{\vec{\kappa}}(\nu) \sinh[\kappa(\zeta - \nu)]\, d\nu \qquad \text{(II-13a)}$$

and from (II-9b) and (II-11c)

$$\phi_{\vec{\kappa}}^{\pm} = C_{\vec{\kappa}}^{\pm}\, e^{\mp\kappa\zeta} \quad . \qquad \text{(II-13b)}$$

Using the boundary conditions (II-11a) and (II-11b), the constants $A_{\vec{\kappa}}$, $B_{\vec{\kappa}}$, and $C_{\vec{\kappa}}^{\pm}$ are found to be

$$A_{\vec{\kappa}} = -\frac{2\pi}{\kappa} \int_0^1 \rho_{\vec{\kappa}}(\zeta)\, e^{-\kappa\zeta}\, d\zeta + \frac{2\pi}{\kappa}\, e^{-\kappa}\, \vec{i}_z \cdot \vec{m}_{\vec{\kappa}}(1)$$

$$B_{\vec{\kappa}} = -\frac{2\pi}{\kappa} \int_1^0 \rho_{\vec{\kappa}}(\zeta)\, e^{\kappa\zeta}\, d\zeta - \frac{2\pi}{\kappa}\, e^{-\kappa}\, \vec{i}_z \cdot \vec{m}_{\vec{\kappa}}(-1)$$

$$C_{\vec{\kappa}}^{\pm} = -\frac{2\pi}{\kappa} \int_{-1}^1 \rho_{\vec{\kappa}}(\zeta)\, e^{\pm\kappa\zeta}\, d\zeta + \frac{2\pi}{\kappa}\, \vec{i}_z \cdot \left[\vec{m}_{\vec{\kappa}}(1)\, e^{\pm\kappa} - \vec{m}_{\vec{\kappa}}(-1)\, e^{\mp\kappa}\right]$$

so that

$$\phi_{\vec{\kappa}}(\zeta) = -\frac{2\pi}{\kappa} \left\{ \int_{-1}^1 \rho_{\vec{\kappa}}(\nu)\, e^{-\kappa|\nu - \zeta|}\, d\nu - \left[\vec{i}_z \cdot \vec{m}_{\vec{\kappa}}(\nu)\, e^{-\kappa|\nu-\zeta|}\right]\Bigg|_{\nu=-1}^{1} \right\} \qquad \text{(II-14a)}$$

and

$$\phi_{\vec{\kappa}}^{\pm}(\zeta) = -\frac{2\pi}{\kappa} \left\{ \int_{-1}^1 \rho_{\vec{\kappa}}(\nu)\, e^{\pm\kappa\nu}\, d\nu - \left[\vec{i}_z \cdot \vec{m}_{\vec{\kappa}}(\nu)\, e^{\pm\kappa\nu}\right]\Bigg|_{\nu=-1}^{1} \right\} e^{\mp\kappa\zeta} \quad . \qquad \text{(II-14b)}$$

A general answer to the question "Given the magnetization distribution in a thin film, what is the magnetostatic field?" is provided by (II-14a, b), since

$$\vec{H} = \sum_{\vec{\kappa}} \vec{h}_{\vec{\kappa}}(\zeta)\, e^{i\vec{\kappa}\cdot\vec{\sigma}} = -\nabla\phi \qquad \text{(II-15a)}$$

with

$$\vec{h}_{\vec{\kappa}}(\zeta) = -i\vec{\kappa}\phi_{\vec{\kappa}}(\zeta) - \vec{i}_z \phi_{\vec{\kappa}}'(\zeta) \qquad \text{(II-15b)}$$

inside the film, and the analogous expression outside the film. However, in the cases we shall be dealing with, $\vec{M}(\vec{\sigma})$ is not given but must be determined statically (by either minimizing the total energy or requiring that the torque $\vec{T}_{eff} = \vec{M} \times \vec{H}_{eff}$ vanish everywhere) or dynamically (from the equation of motion (I-2) for $\vec{M}$, namely $\partial\vec{M}/\partial t = -\gamma\vec{T}_{eff}$). Using the potential (II-14a) will then lead to an integro-differential equation for $\vec{m}_{\vec{\kappa}}(\zeta)$ which might be solved by approximate methods. However, a much simpler procedure – one which is readily applicable to a large class of problems and through which considerable physical insight can be obtained – is to assume that $\vec{m}_{\vec{\kappa}}$ is independent of ζ. This assumption, which we call the thin-film approximation (TFA), is good providing the Fourier components $\vec{m}_{\vec{\kappa}}(\zeta)$ are slowly varying across the film thickness. It will be shown later that for thin enough films, components for which the TFA is not appropriate have very high exchange energies and are, therefore, only weakly excited.

Using the TFA in (II-14a, b), we find

$$\phi_{\vec{\kappa}}(\zeta) = \frac{4\pi}{\kappa}\left[-\frac{i\vec{\kappa}}{\kappa}\,(1 - e^{-\kappa}\cosh\kappa\zeta) + \vec{i}_z\, e^{-\kappa}\sinh\kappa\zeta\right] \cdot \vec{m}_{\vec{\kappa}} \tag{II-16a}$$

$$\phi^{\pm}_{\vec{\kappa}}(\zeta) = \frac{4\pi}{\kappa}\, e^{\mp\kappa\zeta}\sinh\kappa\left(-i\frac{\vec{\kappa}}{\kappa} \pm \vec{i}_z\right) \cdot \vec{m}_{\vec{\kappa}} \quad . \tag{II-16b}$$

The Fourier components of the magnetostatic field are then found from (II-12a-c), (II-15a, b), and (II-16a, b) to be

$$\vec{h}_o = -4\pi\,\vec{i}_z\vec{i}_z \cdot \vec{m}_o \tag{II-17a}$$

and

$$\underset{(\vec{\kappa}\neq 0)}{\vec{h}_{\vec{\kappa}}(\zeta)} = -4\pi\left\{\frac{\vec{\kappa}}{\kappa}\left[\frac{\vec{\kappa}}{\kappa}\,(1 - e^{-\kappa}\cosh\kappa\zeta) + i\,\vec{i}_z\, e^{-\kappa}\sinh\kappa\zeta\right]\right.$$

$$\left. + \vec{i}_z\, e^{-\kappa}\left(i\frac{\vec{\kappa}}{\kappa}\sinh\kappa\zeta + \vec{i}_z\cosh\kappa\zeta\right)\right\} \cdot \vec{m}_{\vec{\kappa}} \tag{II-17b}$$

inside the film, and

$$\vec{h}^{\pm}_o = 0 \tag{II-17c}$$

$$\vec{h}^{\pm}_{\vec{\kappa}}(\zeta) = -4\pi\, e^{\mp\kappa\zeta}\sinh\kappa\left[\frac{\vec{\kappa}}{\kappa}\left(\frac{\vec{\kappa}}{\kappa} \pm i\,\vec{i}_z\right) + \vec{i}_z\left(\pm i\frac{\vec{\kappa}}{\kappa} - \vec{i}_z\right)\right] \cdot \vec{m}_{\vec{\kappa}} \tag{II-17d}$$

above and below the film. Note that (II-17a) and (II-17c) are also valid exactly, i.e., for $\vec{m}_o = \vec{m}_o(\zeta)$.

The field inside the film is of primary interest, since only this field contributes to the torque on $\vec{M}$. The TFA magnetostatic torque may be expanded in a Fourier series

$$\vec{T}(\vec{\sigma}) = \vec{M}(\xi,\eta) \times \vec{H}(\vec{\sigma}) = \sum_{\vec{\kappa}} \vec{t}_{\vec{\kappa}}(\zeta)\, e^{i\vec{\kappa}\cdot\vec{\sigma}} \tag{II-18a}$$

where the components of $\vec{T}$ are given by

$$\vec{t}_o(\zeta) = \vec{m}_o \times \vec{h}_o + \sum_{\vec{\kappa} \neq 0} \vec{m}_{-\vec{\kappa}} \times \vec{h}_{\vec{\kappa}}(\zeta) \tag{II-18b}$$

$$\vec{t}_{\vec{\kappa}}(\zeta) = \vec{m}_o \times \vec{h}_{\vec{\kappa}}(\zeta) + \vec{m}_{\vec{\kappa}} \times \vec{h}_o + \sum_{\vec{\kappa}' \neq \vec{\kappa}, 0} \vec{m}_{\vec{\kappa}-\vec{\kappa}'} \times \vec{h}_{\vec{\kappa}'}(\zeta) \quad . \tag{II-18c}$$

In keeping with the TFA, it is the ζ-average of the torque components which is of physical interest. Since the $\vec{h}_{\vec{\kappa}}(\zeta)$ are the only ζ-dependent factors, we may replace $\vec{h}_{\vec{\kappa}}(\zeta)$ by

$$\vec{h}_{\vec{\kappa}} = \frac{1}{2} \int_{-1}^{1} \vec{h}_{\vec{\kappa}}(\zeta)\, d\zeta \quad . \tag{II-19}$$

Then from (II-17b) we have our final result for the magnetostatic field components

$$\vec{h}_{\vec{\kappa}} = -4\pi \left[\frac{\vec{\kappa}\vec{\kappa}}{\kappa^2} \tilde{\chi}(\kappa) + \vec{i}_z \vec{i}_z \chi(\kappa) \right] \cdot \vec{m}_{\vec{\kappa}} \tag{II-20a}$$

where

$$\chi(\kappa) = \kappa^{-1} e^{-\kappa} \sinh \kappa \tag{II-20b}$$

$$\tilde{\chi}(\kappa) = 1 - \chi(\kappa) \quad . \tag{II-20c}$$

Note that since $\chi(0) = 1$ and $\tilde{\chi}(0) = 0$, (II-20a) holds for $\vec{\kappa} = 0$ also.

We next examine $\vec{h}_{\vec{\kappa}}$ in the limits $\kappa >> 1$ (wavelength small compared with film thickness) and $\kappa << 1$ (wavelength long compared with film thickness). For the first limit,

$$\chi(\kappa) = \frac{1}{2\kappa} + \ldots \tag{II-21a}$$

so that

$$\tilde{\chi}(\kappa) = 1 - \frac{1}{2\kappa} + \ldots \tag{II-21b}$$

and

$$\vec{h}_{\vec{\kappa}} \simeq -4\pi \left(\frac{\vec{\kappa}\vec{\kappa}}{\kappa^2} + \frac{\vec{i}_z \vec{i}_z}{2\kappa} \right) \cdot \vec{m}_{\vec{\kappa}} \quad (\kappa >> 1) \quad . \tag{II-21c}$$

The first term of (II-21c) is the usual infinite medium magnetostatic field from volume poles and follows immediately from (II-1a-c) with periodic boundary conditions. The second term is a small correction due to the fact that surface poles do not quite cancel themselves out for a film of finite thickness.

For the second limit,

$$\chi(\kappa) = 1 - \kappa + \frac{2}{3} \kappa^2 + \ldots \tag{II-22a}$$

$$\tilde{\chi}(\kappa) = \kappa - \frac{2}{3} \kappa^2 + \ldots \tag{II-22b}$$

so that

$$\vec{h}_{\vec{\kappa}} \simeq -4\pi\left(\frac{\vec{\kappa}\vec{\kappa}}{\kappa} + \vec{i}_z\vec{i}_z\right) \cdot \vec{m}_{\vec{\kappa}} \qquad (\kappa \ll 1) \quad . \tag{II-22c}$$

The first term of (II-22c), written with non-normalized wave vector $\vec{k} = \vec{\kappa}/L$, is

$$\vec{h}_{\vec{k}} = -4\pi L \frac{\vec{k}\vec{k}}{k} \cdot \vec{m}_{\vec{k}} \quad . \tag{II-23}$$

This is just the planar magnetostatic field in the long wavelength limit, calculated by some authors.[26] It is the infinite medium magnetostatic field from volume poles attenuated by a factor $kL \ll 1$; the reason for this attenuation is simply the absence of volume poles from the regions above and below the film. The second term in (II-22c) is the normal demagnetizing field from surface poles.

To summarize, the magnetostatic field components are given in the TFA by

$$\boxed{\begin{aligned} \vec{h}_o &= -4\pi\, \vec{i}_z\vec{i}_z \cdot \vec{m}_o \\ \vec{h}_{\vec{k}} &= -4\pi\left[\frac{\vec{k}\vec{k}}{k^2}\,\tilde{\chi}(kL) + \vec{i}_z\vec{i}_z\,\chi(kL)\right] \cdot \vec{m}_{\vec{k}} \end{aligned}} \qquad \begin{aligned} &\text{(II-24a)} \\ &\text{(II-24b)} \end{aligned}$$

where

$$\vec{H}(\vec{r}) = \vec{h}_o + \sum_{\vec{k}\neq 0} \vec{h}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \tag{II-24c}$$

and χ and $\tilde{\chi}$ are given by (II-20b) and (II-20c). Equation (II-24a) is the uniform demagnetizing field of an oblate spheriod in the zero thickness limit. The first term in (II-24b) is from volume poles, with an attenuation factor $\tilde{\chi}$ due to the presence of finite boundaries. The second term in (II-24b) is from the surface poles, with an attenuation factor χ due to finite wavelength and therefore some cancellation of these poles. We now have the thin-film magnetostatic field in a simple form which may be used to calculate the static magnetization configuration (Sec. III) and interactions in large-angle magnetization rotations (Sec. V), and as a by-product we can also obtain the planar mode spectrum (including exchange fields) for all wavelengths (Sec. V).

III. STATIC EQUILIBRIUM: MAGNETIZATION RIPPLE

A. INTRODUCTION: EXPERIMENTAL BACKGROUND

To find the transient response to an external field of the magnetization $\vec{M}(\vec{r}, t)$ of a ferromagnet, the initial state $\vec{M}(\vec{r}, 0)$ must be known, which in our formulation means specifying the initial values of the Fourier components $\vec{m}_{\vec{k}}(t)$ for all $\vec{k}$. The most important sources of disturbances $\vec{m}_{\vec{k}}(0)$ ($\vec{k} \neq 0$) are thermal agitation and inhomogeneities. Thermally excited spin waves, which are present in any ferromagnet at a finite temperature, and which result in the well-known decrease of saturation magnetization M_o with increasing temperature, will be important dynamically if they become unstable. However, it will be shown in Sec. V that while such instabilities do occur,[27] their growth times are always longer than the time for coherent switching, so that they have no significant first order effects.

Inhomogeneities are present to some extent in any ferromagnetic crystal. However, in a polycrystalline sample, in which we are interested in this work, particularly large dispersive effects are to be expected, both from crystalline anisotropy forces and from strains between crystallites. In this section we calculate the equilibrium dispersion-induced components $\vec{m}_{\vec{k}}(0)$ [henceforth the argument (0) will be dropped] for a polycrystalline thin film (<2000 Å), represented by a two-dimensional model. (The model, however, is readily capable of extension to three dimensions to treat bulk polycrystals and thick films.)

The solutions we will find are not only of importance in specifying initial values in the dynamic problem, but are also of interest in their own right. Before proceeding with the model, it will be useful to review the pertinent experimental evidence concerning the equilibrium magnetization configuration in thin films. In 1960 Fuller and Hale,[28] using a defocused electron microscope, observed the Lorentz deflection of electrons by the magnetic induction field $\vec{B} \simeq 4\pi\vec{M}(\vec{r})$, and discovered a wave-like magnetic fine structure which they called magnetization ripple. This discovery has been confirmed by subsequent investigators,[29,30] and an example is shown in Fig. 2.

Fuller and Hale described two basic ripple components: longitudinal (LMR), in which $\vec{M} = \vec{M}(x)$ ($\vec{k} \parallel \vec{i}_x$), and lateral or transverse (TMR), in which $\vec{M} = \vec{M}(y)$ ($\vec{k} \parallel \vec{i}_y$), where the mean magnetization $\vec{m}_o$ lies along the x-axis of a Cartesian coordinate system with the z-axis normal to the film plane. These two ripple components are shown schematically in Fig. 2. Since the volume pole density is very small for LMR ($\nabla \cdot \vec{M} \simeq 0$) but not for TMR ($\nabla \cdot \vec{M} \neq 0$), and since the two are equivalent with respect to exchange, anisotropy, and Zeeman forces, the main contribution to the magnetization ripple should be LMR. Fuller and Hale thus interpreted the fine-structure lines they observed as the loci of constant $\vec{M}$, orthogonal to $\vec{m}_o$. They found a mean wavelength of $\sim 2 \times 10^{-4}$ cm and a mean amplitude of $\sim 10^{-2}$ radian.

Fuchs[29] has suggested that the origin of ripple is to be found in crystalline anisotropy forces, which vary randomly in direction from crystallite to crystallite (crystallite size $\simeq 10^{-6}$ cm; crystalline anisotropy energy $\simeq 10^4$ erg/cm^3). The magnetization does not follow these local wanderings of the direction of minimum anisotropy energy but, because of exchange coupling, which tends to straighten the path of $\vec{M}$, follows the mean easy axis averaged over a number of crystallites. For TMR, this number is greatly increased by magnetostatic coupling, so that the amplitude of TMR is much less than that of LMR. The mean magnetization direction is determined by the uniaxial $\vec{M}$-induced anisotropy and the external field.

In addition to crystalline anisotropy, there will also be, as just mentioned, random local magnetostrictive forces due to inhomogeneous strains between crystallites (or clumps of crystallites). Stresses (for example, surface tension) between adjacent crystallites on the substrate to which the film is bonded result in planar strains; z-directed stresses, on the other hand, may be for the most part relieved, since the free upper surface of the film imposes no comparable constraint. Thus we expect the local magnetostrictive anisotropy axes to lie nearly in the film plane and to be randomly oriented in this plane. Experimental evidence for such an anisotropy is provided by oblique-incidence Permalloy films,[31] which are deposited by a vapor beam at a fixed angle Θ_i to the substrate normal. (In normal films $\Theta_i = 0$.) In such films, crystallites form chains perpendicular to the incident beam by means of a self-shadowing mechanism, so that strain axes are not random but tend to lie along these chains. As a result of both anisotropic strain and shape anisotropy of the chains, these films possess a large over-all uniaxial anisotropy $K_u \simeq 10^5\ \mathrm{erg/cm^3}$, depending on Θ_i and film composition. Shape effects may be separated from strain effects by considering the compositional dependence of K_u; from this, one may estimate random anisotropy energy in normal films of $\sim 5 \times 10^4\ \mathrm{erg/cm^3}$ for a TPF.

The relative roles played by crystalline and strain anisotropies in inducing magnetization ripple have not been firmly established experimentally. In what follows we shall neglect crystalline anisotropy since strain effects are probably greater and strain anisotropy is easier to handle analytically. However, consideration of crystalline anisotropy would not alter the results in any significant qualitative way. We assume a random local magnetostrictive anisotropy which is planar, uniaxial, and constant across the film thickness and, using a simple model for a polycrystalline film, proceed to calculate the ripple spectrum in static equilibrium.

B. RANDOM ANISOTROPY MODEL

Our model for a film is a planar array of N right cylindrical cells of height 2L (= film thickness) and cross-sectional areas $a_m (m = 1, 2, \ldots N)$. The a_m are assumed to be random variables with ensemble average $\bar{a}$, and the cross-sectional shapes are also assumed to vary randomly. By this we mean precisely that the intersections of any line in the film plane with the cell boundaries occur at random (uncorrelated) intervals along the line (see Fig. 3).

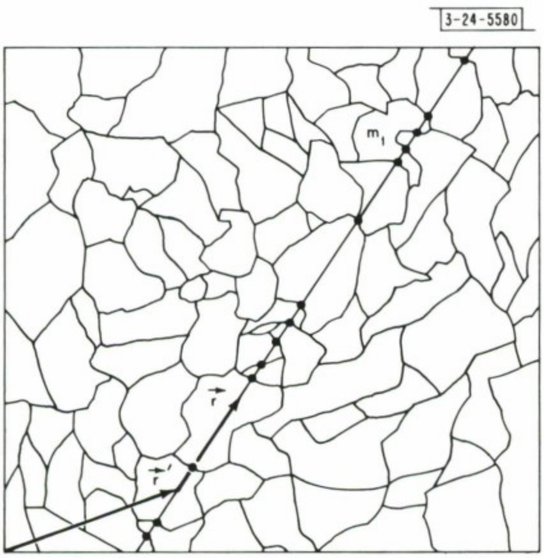

Fig. 3. Random cell model showing construction used to find autocorrelation function $c_o(\vec{r})$.

Furthermore, the mean number of such intervals per unit length, r_o^{-1}, is assumed to be the same for all lines in the plane; i.e., cell shapes are on the average isotropic and cell sizes and shapes are on the average homogeneous. The film plane is of area $S = N\bar{a}$, and we shall eventually pass to the limit $N \to \infty$ ($\bar{a}$ fixed).

Each cell has a magnetic anisotropy which is the sum of two terms: a uniaxial $\vec{M}$-induced anisotropy, common to all cells, with energy density K_o and easy axis the x-axis, and a uniaxial strain-induced anisotropy with energy density K_{1m} and axis in the film plane at an angle α_m to the x-axis, where α_m may take on any value between $-\pi/2$ and $\pi/2$ with equal probability. We further assume zero correlation among the quantities a_m, K_{1m}, and α_m, and that the variations in K_{1m} and α_m are uncorrelated from one cell to the next. This addition of uniform and random uniaxial anisotropies has been used before to explain some film properties,[32] but a model was lacking and exchange and magnetostatic interactions were neglected; in the present work interactions play a vital role. Rother[33] has analyzed a model similar to ours; however, in his model, $K_o = 0$ and the cells have equal square cross sections. He has included the effects of exchange interactions, and magnetostatic interactions as a perturbation, but his results are quite different from ours for reasons which will be made clear at the end of this section.

The anisotropy energy density at any point in the film is given by an obvious generalization of (I-6)

$$E_a(\vec{r}) = -M_o^{-2}\{K_o[\vec{M}(\vec{r})\cdot\vec{i}_x]^2 + K_1(\vec{r})[\vec{M}(\vec{r})\cdot\vec{\alpha}(\vec{r})]^2\} \qquad \text{(III-1)}$$

where $\vec{\alpha}(\vec{r})$ is a unit vector at angle α to the x-axis; for $\vec{r}$ in the m^{th} cell, $K_1(\vec{r}) = K_{1m}$ and $\alpha(\vec{r}) = \alpha_m$. The anisotropy field is then

$$\vec{H}_a = -\nabla_{\vec{M}} E_a = 2M_o^{-2}[K_o(\vec{i}_x - \vec{M}\,\vec{i}_x\cdot\vec{M}M_o^{-2})\,\vec{i}_x\cdot\vec{M} + K_1(\vec{\alpha} - \vec{M}\vec{\alpha}\cdot\vec{M}M_o^{-2})\,\vec{\alpha}\cdot\vec{M}]$$

so that the anisotropy torque is given by

$$\vec{T}_a = \vec{M}\times\vec{H}_a = 2M_o^{-2}\vec{M}\times(K_o\vec{i}_x\vec{i}_x + K_1\vec{\alpha}\vec{\alpha})\cdot\vec{M} \quad . \qquad \text{(III-2)}$$

Since we shall be interested only in the case of external fields in the film plane, $\vec{M}$ must lie in this plane in static equilibrium† and is, therefore, determined by the angle $\varphi(\vec{r})$ between $\vec{M}$ and the x-axis. Equation (III-2) then becomes

$$\vec{T}_a = \vec{i}_z[-K_o \sin 2\varphi + K_1 \sin 2(\alpha - \varphi)] \quad . \qquad \text{(III-2-1)}$$

We next separate the spatially varying part of φ, writing

$$\varphi(\vec{r}) = \varphi_o + \delta\varphi(\vec{r}) \qquad \text{(III-3a)}$$

where

$$\varphi_o = \langle\varphi(\vec{r})\rangle \equiv \frac{1}{S}\int \varphi(\vec{r})\, d^2\vec{r} \qquad \text{(III-3b)}$$

and from (I-1a)

$$|\delta\varphi| << 1 \quad . \qquad \text{(III-3c)}$$

† This may not be true if there is a strong easy axis normal to the film plane; but in that case, deviations of $\vec{M}$ from its mean direction are large,[34] and thus beyond the scope of this linearized treatment.

Inserting (III-3a) in (III-2-1) and dropping terms of order $(\delta\varphi)^2$ and higher, we find

$$\vec{T}_a = \vec{i}_z \{-K_o \sin 2\varphi_o + P(\vec{r}) - 2\delta\varphi(\vec{r}) [K_o \cos 2\varphi_o + Q(\vec{r})]\} \tag{III-4}$$

where

$$P(\vec{r}) = K_1(\vec{r}) \sin 2 [\alpha(\vec{r}) - \varphi_o] \tag{III-5a}$$

$$Q(\vec{r}) = K_1(\vec{r}) \cos 2 [\alpha(\vec{r}) - \varphi_o] \quad . \tag{III-5b}$$

The functions P and Q contain all effects of the random anisotropy field. P may be thought of as a random perturbing force on $\delta\varphi$, and Q as a random restoring force. The ensemble averages of these functions vanish, since

$$\overline{P} = \overline{K_1 \sin 2(\alpha - \varphi_o)}$$

$$= \overline{K_1}\ \overline{\sin 2(\alpha - \varphi_o)} \quad \text{(since } K_{1m} \text{ and } \alpha_m \text{ are uncorrelated)}$$

$$= K_1 \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \sin 2(\alpha - \varphi_o)\, d\alpha = 0$$

and similarly for Q.

We now expand $\delta\varphi$, P, and Q in Fourier series

$$\delta\varphi(\vec{r}) = \sum_{\vec{k} \neq 0} \varphi_{\vec{k}}\, c^{i\vec{k}\cdot\vec{r}} \tag{III-6a}$$

$$P(\vec{r}) = \sum_{\vec{k}} p_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \tag{III-6b}$$

$$Q(\vec{r}) = \sum_{\vec{k}} q_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} \tag{III-6c}$$

where the wave vector $\vec{k}$ takes on values imposed by periodic boundary conditions at the film edge $x = \pm L_x$, $y = \pm L_y$:

$$\vec{k} = \pi\left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, 0\right) \quad (n_x, n_y = 0, \pm 1, \ldots) \tag{III-7}$$

with the condition

$$k\ell \ll 1$$

(ℓ = lattice constant) insuring the validity of our continuum treatment of the magnetization. The components of (III-4) then become

$$(\vec{t}_o)_a = \vec{i}_z\left(-K_o \sin 2\varphi_o + p_o - 2 \sum_{\vec{k}' \neq 0} q_{-\vec{k}'} \varphi_{\vec{k}'}\right) \tag{III-8a}$$

$$(\vec{t}_{\vec{k}})_a = \vec{i}_z\left(p_{\vec{k}} - 2K_o \cos 2\varphi_o \varphi_{\vec{k}} - 2 \sum_{\vec{k}' \neq 0} q_{\vec{k}-\vec{k}'} \varphi_{\vec{k}'}\right) \quad . \tag{III-8b}$$

These expressions will be combined with the remaining torques acting on $\vec{M}$ (exchange, magnetostatic, and Zeeman), to determine the ripple components $\varphi_{\vec{k}}$. In particular, we will be interested in the "power" spectrum $\overline{|\varphi_{\vec{k}}|^2}$, and to find this ensemble average will require the power spectrum $\overline{|f_{\vec{k}}|^2}$ of a random function F of the form

$$F(\vec{r}) = F_m \quad \text{for} \quad \vec{r} \text{ in } m^{th} \text{ cell} \tag{III-9a}$$

$$\overline{F(\vec{r})} = 0 \tag{III-9b}$$

where F may be P or Q, and

$$F(\vec{r}) = \sum_{\vec{k}} f_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} \quad (\vec{k} \text{ in film plane}) \quad . \tag{III-9c}$$

To accomplish this we first find the autocorrelation function

$$c_o(\vec{r}) = \frac{1}{S}\int \overline{F(\vec{r}' + \vec{r})\, F(\vec{r}')}\, d^2\vec{r}' \tag{III-10}$$

by a generalization of a method due to Kenrick.[35] Consider a straight line in the film plane defined by the vector $\vec{r}$, with origin at an arbitrary point $\vec{r}'$ (see Fig. 3). We first determine the probability $P_o(\vec{r}, \vec{r}')$ that no cell boundary intersects this line between $\vec{r}'$ and $\vec{r}' + \vec{r}$. From homogeneity and isotropy, respectively, $P_o(\vec{r}, \vec{r}') = P_o(\vec{r}) = P_o(r)$. Now the probability that there are no cell boundaries between 0 and $r + \Delta r$ is equal to the probability that there are none between 0 and r, multiplied by the probability that there are none in the interval Δr. This latter probability, in the limit $\Delta r \to 0$, is $1 - r_o^{-1}\Delta r$ (where r_o^{-1} is the density of cell boundaries along any line). Thus

$$\lim_{\Delta r\to 0} P_o(r)(1 - r_o^{-1}\Delta r) = \lim_{\Delta r\to 0} P_o(r + \Delta r)$$

$$= P_o(r) + \Delta r P_o'(r) + \ldots$$

or

$$P_o'(r) = -r_o^{-1} P_o(r)$$

which has the solution

$$P_o(r) = e^{-r/r_o} \tag{III-11}$$

[since $P_o(0) = 1$].

Since $F(\vec{r})$ is a random function with zero mean, the integrand of (III-10) vanishes unless $\vec{r}' + \vec{r}$ and $\vec{r}'$ are in the same cell, in which case it is equal to $\overline{F^2}$. Thus we have the result

$$c_o(\vec{r}) = c_o(r) = \overline{F^2}\, e^{-r/r_o} \quad . \tag{III-12}$$

This expression is somewhat in error for two reasons. First, it ignores the effects of finite boundaries, which would result in a correction of order N^{-1} and may therefore be neglected. Second, it does not consider the possibility that because of a concave cell boundary, both $\vec{r}$

and $\vec{r}'$ may lie in the same cell although the straight line connecting them passes through one or more other cells (as in cell m_1, Fig. 3). However, unless the cells are serpentine, which is not at all plausible physically since crystallites grow outward in all directions from a nucleus, contributions from such double crossings (and triple, etc.) should be very small and can safely be ignored.

The power spectrum is just the Fourier transform of the autocorrelation function:

$$\overline{|f_{\vec{k}}|^2} = \frac{1}{S}\int c_o(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^2\vec{r}$$

$$= \frac{\overline{F^2}}{S}\left\{\int_o^\infty e^{-r/r_o}\, rdr \int_o^{2\pi} e^{-ikr\cos\theta}\, d\theta - \iint_{\tilde{S}} \exp\left[-\frac{\sqrt{x^2+y^2}}{r_o} - i(k_x x + k_y y)\right] dxdy\right\} \qquad \text{(III-13)}$$

where $\tilde{S}$ is the region $|x| > L_x$, $|y| > L_y$. The first term in the bracket in (III-13) is

$$2\pi \int_o^\infty e^{-r/r_o} J_o(kr)\, rdr = 2\pi r_o^2 (1 + k^2 r_o^2)^{-3/2} \quad .$$

The second term vanishes as L_x, $L_y \to \infty$:

$$\left|\iint_{\tilde{S}} \exp\left[-\frac{\sqrt{x^2+y^2}}{r_o} - i(k_x x + k_y y)\right] dxdy\right| \leqslant \iint_{\tilde{S}} \exp\left[-\frac{\sqrt{x^2+y^2}}{r_o}\right] dxdy$$

$$< \int_{L_s}^\infty e^{-r/r_o}\, rdr \int_o^{2\pi} d\theta = 2\pi r_o (L_s + r_o)\, e^{-L_s/r_o}$$

which goes to zero as $L_s \to \infty$, where L_s is the smaller of L_x and L_y. Thus we have the main result of this section

$$\overline{|f_k|^2} = \frac{\overline{F^2}}{S}\, 2\pi r_o^2 (1 + k^2 r_o^2)^{-3/2} \quad . \qquad \text{(III-14)}$$

It may be helpful to know the mean cell boundary spacing r_o in terms of the cell areas a_m; this is most easily found by direct evaluation of $\overline{|f_o|^2}$ and comparison with (III-14). The $\vec{k} = 0$ component of F is

$$f_o = \frac{1}{S}\int F(\vec{r})\, d^2\vec{r}$$

$$= \frac{1}{S}\sum_{m=1}^{N} F_m a_m$$

so that

$$\overline{|f_o|^2} = \frac{1}{S^2} \sum_{m=1}^{N} \sum_{n=1}^{N} \overline{F_m F_n a_m a_n}$$

$$= \frac{\overline{F^2}}{S^2} \sum_{m=1}^{N} \overline{a_m^2} \qquad \text{(since } \overline{F_m F_n} = \overline{F^2}\delta_{m,n}\text{)}$$

$$= \frac{\overline{F^2}}{S} \frac{\overline{a^2}}{\overline{a}} \qquad \text{(since } N = S/\overline{a}\text{)}$$

giving

$$r_o = \left(\frac{\overline{a^2}}{2\pi\overline{a}}\right)^{1/2} \quad . \tag{III-15}$$

Finally, from (III-5a, b),

$$\overline{P^2} = \overline{K_1^2}\ \overline{\sin^2 2\,[\alpha(\vec{r}) - \varphi_o]} = \frac{1}{2}\overline{K_1^2} = \overline{Q^2} \quad . \tag{III-16}$$

C. MAGNETIZATION RIPPLE SPECTRUM

The remaining contributions to the torque are now collected. We consider the case of a steady, uniform, external magnetic field $\vec{H}$ in the film plane at an angle β to the x-axis (see Fig. 4). The Zeeman torque, in the linear approximation, is then

$$\vec{T}_h = \vec{M} \times \vec{H}$$

$$= \vec{i}_z M_o H\,[\sin(\beta - \varphi_o) - \delta\varphi\,\cos(\beta - \varphi_o)] \tag{III-17a}$$

with Fourier components

$$(\vec{t}_o)_h = \vec{i}_z M_o H \sin(\beta - \varphi_o) \tag{III-17b}$$

$$(\vec{t}_{\vec{k}})_h = -\vec{i}_z M_o H \cos(\beta - \varphi_o)\,\varphi_{\vec{k}} \quad . \tag{III-17c}$$

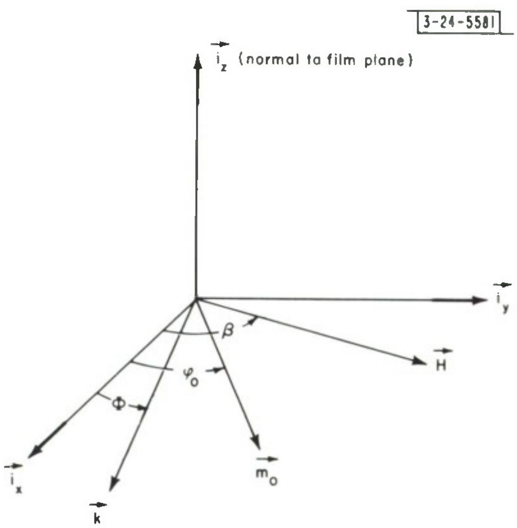

Fig. 4. Coordinate system for static case.

The exchange field is given by (I-4)

$$\vec{H}_e = \frac{2A}{M_o^2} \nabla^2 \vec{M}(\vec{r}) \simeq \frac{2A}{M_o} (-\vec{i}_x \sin\varphi_o + \vec{i}_y \cos\varphi_o) \nabla^2 \delta\varphi \tag{III-18a}$$

where A is the exchange constant. This field exerts an exchange torque

$$\vec{T}_e = \vec{M} \times \vec{H}_e = \vec{i}_z 2A\nabla^2 \delta\varphi \tag{III-18b}$$

with Fourier components

$$(\vec{t}_o)_e = 0 \tag{III-18c}$$

$$\left(\vec{t}_{\vec{k}}\right)_e = -\vec{i}_z 2Ak^2 \varphi_{\vec{k}} \quad . \tag{III-18d}$$

Finally we have the magnetostatic field with Fourier components in the TFA given by (II-24a, b), which become

$$(\vec{h}_o)_m = 0 \tag{III-19a}$$

$$\left(\vec{h}_{\vec{k}}\right)_m = -4\pi\tilde{\chi}(kL) \frac{\vec{k}\vec{k} \cdot \vec{m}_{\vec{k}}}{k^2} \tag{III-19b}$$

where

$$\tilde{\chi}(\kappa) = 1 - \kappa^{-1} e^{-\kappa} \sinh\kappa = \kappa - \frac{2}{3}\kappa^2 + \ldots \quad . \tag{III-20}$$

The linearized torque components are then

$$(\vec{t}_o)_m = 0 \tag{III-21a}$$

$$\left(\vec{t}_{\vec{k}}\right)_m = -\vec{i}_z 4\pi M_o^2 \tilde{\chi}(kL) \sin^2(\Phi - \varphi_o) \varphi_{\vec{k}} \tag{III-21b}$$

where $\vec{k}$ lies in the film plane at an angle Φ to the x-axis (see Fig. 4).

The equilibrium value of $\delta\varphi$ is determined by the condition that the effective torque vanishes, or

$$\vec{T}_{eff}(\vec{r}) = \vec{T}_a + \vec{T}_h + \vec{T}_e + \vec{T}_m = \sum_{\vec{k}} \left(\vec{t}_{\vec{k}}\right)_{eff} e^{i\vec{k}\cdot\vec{r}} = 0 \quad . \tag{III-22}$$

The Fourier components of (III-22) vanish independently, so that from (III-8a), (III-17b), (III-18c), and (III-21a)

$$-K_o \sin 2\varphi_o + p_o - 2 \sum_{\vec{k}\neq 0} q_{-\vec{k}} \varphi_{\vec{k}} + M_o H \sin(\beta - \varphi_o) = 0 \quad . \tag{III-23a}$$

For $\vec{k} \neq 0$, from (III-8b), (III-17c), (III-18d), and (III-21b)

$$-2K_o \cos 2\varphi_o \varphi_{\vec{k}} + p_{\vec{k}} - 2 \sum_{\vec{k}' \neq 0} q_{\vec{k}-\vec{k}'} \varphi_{\vec{k}'} - [M_o H \cos(\beta - \varphi_o) + 2Ak^2$$

$$+ 4\pi M_o^2 \tilde{\chi} \sin^2(\Phi - \varphi_o)] \varphi_{\vec{k}} = 0 \quad . \qquad \text{(III-23b)}$$

We solve (III-23a, b) by an iterative procedure, first assuming $\varphi_{\vec{k}} = 0$ for all $\vec{k} \neq 0$ and solving for φ_o. Equation (III-23a) becomes

$$\frac{1}{2} \sin 2\varphi_o - h \sin(\beta - \varphi_o) = \frac{1}{2} \frac{p_o}{K_o} \qquad \text{(III-24)}$$

where the reduced field h is given by

$$h = \frac{H}{H_K} = \frac{HM_o}{2K_o} \quad . \qquad \text{(III-25)}$$

The term p_o is the magnitude of the (spatial) average torque on $\vec{M}$ due to random anisotropy forces, and is only nonzero because for any given ensemble of values of α_m, a_m, and K_{1m} there is a small but finite probability that this average does not vanish. Of course the ensemble average $\overline{p}_o = 0$.

The mean square value of the right side of (III-24), obtained with the help of (III-14) and (III-16) is

$$\frac{\overline{|p_o|^2}}{4K_o^2} = \frac{\Gamma^2}{8} \frac{2\pi r_o^2}{S} \qquad \text{(III-26)}$$

where Γ is the ratio of the rms value of the random anisotropy energy to the uniform anisotropy energy, or

$$\Gamma^2 = \frac{\overline{K_1^2}}{K_o^2} \quad . \qquad \text{(III-27)}$$

Since from (III-26) the right side of (III-24) vanishes in the limit $S \to \infty$, we may neglect it in a first approximation to φ_o, obtaining

$$\frac{1}{2} \sin 2\varphi_o - h \sin(\beta - \varphi_o) = 0 \qquad \text{(III-28)}$$

which is the well-known equilibrium equation for the magnetization direction of a uniaxial film.[13] We shall be interested only in three special cases of this equation, for which solutions are

$$\varphi_o = 0 \qquad (\beta = 0,\ h > -1) \qquad \text{(III-29a)}$$

$$\varphi_o = \pi/2 \qquad (\beta = \pi/2,\ h > 1) \qquad \text{(III-29b)}$$

$$\varphi_o = \arcsin h \qquad (\beta = \pi/2,\ |h| < 1) \quad . \qquad \text{(III-29c)}$$

It is easy to see, by examining the average uniaxial anisotropy and Zeeman energies

$$E_o = K_o \sin^2\varphi_o - M_oH\cos(\beta - \varphi_o) \quad , \tag{III-30}$$

that the solutions (III-29a-c) minimize E_o, and are therefore stable.

The next step in the iteration procedure is to assume that there is but one nonvanishing $\varphi_{\vec{k}}(\vec{k} \neq 0)$, and using the value we have just obtained for φ_o, solve (III-23b) for $\varphi_{\vec{k}}$, with $\varphi_{\vec{k}'}(\vec{k}' \neq \vec{k}, 0)$ set equal to zero. This just means that to a first approximation the ripple components $\varphi_{\vec{k}}$ are decoupled. (Note that it is only through the random anisotropy components $q_{\vec{k}-\vec{k}'}$ that $\varphi_{\vec{k}}$ and $\varphi_{\vec{k}'}$ are coupled.) Equation (III-23b) becomes

$$\left[\frac{q_o}{K_o} + \cos 2\varphi_o + h\cos(\beta - \varphi_o) + \frac{Ak^2}{K_o} + \frac{2\pi M_o^2}{K_o}\tilde{\chi}\sin^2(\Phi - \varphi_o)\right]\varphi_{\vec{k}} = \frac{1}{2}\frac{p_{\vec{k}}}{K_o} \quad . \tag{III-31}$$

The term $q_oK_o^{-1}$ in (III-31) may be safely neglected, for the same reason that the term $p_oK_o^{-1}$ was negligible in (III-24), provided

$$\Lambda(\beta, h) = \cos 2\varphi_o + h\cos(\beta - \varphi_o) \neq 0 \tag{III-32}$$

with φ_o determined by (III-28). But Λ is proportional to $\partial^2E_o/\partial\varphi_o^2$ [see (III-30)] so that for stable equilibrium

$$\Lambda > 0 \tag{III-33}$$

and (III-32) is satisfied. The set of values (β_t, h_t) such that

$$\Lambda(\beta_t, h_t) = 0 \tag{III-34}$$

occurs at transitions from stable to unstable equilibrium and are thus threshold angles and fields for irreversible, coherent rotation of $\vec{m}_o$.[13] We may expect serious difficulties in the theory near such thresholds.

For $\Lambda \neq 0$, (III-31) becomes

$$\underset{(\vec{k} \neq 0)}{\varphi_{\vec{k}}} = \frac{1}{2}\frac{p_{\vec{k}}}{K_o\Lambda}\left[1 + \lambda_e^2k^2 + \lambda_m L^{-1}\tilde{\chi}\sin^2(\Phi - \varphi_o)\right]^{-1} \tag{III-35}$$

where we have defined an "exchange length"

$$\lambda_e = \left(\frac{A}{K_o\Lambda}\right)^{1/2} \tag{III-36}$$

and a "magnetostatic length"

$$\lambda_m = \frac{2\pi M_o^2 L}{K_o\Lambda} \quad . \tag{III-37}$$

$\Lambda(\beta, h)$, defined by (III-32), is for cases (a), (b), and (c), respectively, of (III-29)

$$\Lambda(0, h) = h + 1 \qquad (h > -1) \tag{III-38a}$$

$$\Lambda_>(\tfrac{\pi}{2}, h) = h - 1 \qquad (h > 1) \tag{III-38b}$$

$$\Lambda_{<}(\frac{\pi}{2}, h) = 1 - h^2 \qquad (|h| < 1) \quad . \tag{III-38c}$$

The mean square ripple amplitudes then follow from (III-14), (III-16), (III-27), and (III-35):

$$\boxed{\overline{|\varphi_{\vec{k}}|^2} = \frac{\Gamma^2}{8\Lambda^2} \frac{2\pi r_o^2}{S} (1 + r_o^2 k^2)^{-3/2} (1 + g_{\vec{k}})^{-2} \quad (\vec{k} \neq 0)} \tag{III-39}$$

where

$$g_{\vec{k}} = \lambda_e^2 k^2 + \lambda_m L^{-1} \tilde{\chi}(kL) \sin^2(\Phi - \varphi_o) > 0 \quad . \tag{III-40}$$

Equation (III-39), the ripple spectrum, is the main result of Part C. In Appendix B we return to (III-23b), and carrying our iteration procedure one step further show that the first order coupling [meaning the $\varphi_{\vec{k}'}$ are the coupling-free values of (III-35)] introduces a negligible correction to $\overline{|\varphi_{\vec{k}}|^2}$. This gives us some confidence in our method of solution and in the result [(III-39)], which we shall use in Sec. V. In Sec. III-D, however, we examine this result and show that it represents a magnetic fine structure with the characteristics of the observed ripple.

D. DISCUSSION OF RESULT

First, it must be pointed out that even though the mean square Fourier components given by (III-39) are proportional to S^{-1} and therefore vanish in the limit $S \to \infty$, their density in the $\vec{k}$-plane, from (III-7), is

$$\frac{S}{(2\pi)^2} \quad . \tag{III-41}$$

Thus any observable effects of the ripple components are independent of film area S, provided such effects do not become too large for long wavelengths (of the order of L_x or L_y). An example of an observable effect is the mean square magnetization dispersion

$$\delta^2 = \overline{\langle[\varphi(\vec{r}) - \varphi_o]^2\rangle} = \overline{\langle[\delta\varphi(\vec{r})]^2\rangle} \tag{III-42}$$

which we calculate in Sec. III-E. From Parseval's theorem [or from (III-10) and (III-13)]

$$\delta^2 = \sum_{\vec{k} \neq 0} \overline{|\varphi_{\vec{k}}|^2} \tag{III-43}$$

and if the main contribution to this sum is from components with wavelength short compared to L_x and L_y, we may replace it by an integral using the correspondence

$$\sum_{\vec{k}} f_{\vec{k}} \to \frac{S}{(2\pi)^2} \int f(\vec{k})\, d^2\vec{k} \tag{III-44}$$

which follows from (III-41). [$f(\vec{k})$ is just $f_{\vec{k}}$ with $\vec{k}$ considered a continuous variable.] We see then that δ^2 is independent of S, which makes physical sense.

Next we consider the first factor of (III-39):

$$\frac{\Gamma^2}{8\Lambda^2} = \frac{\frac{1}{2}\,\overline{K_1^2}}{[2K_o \cos 2\varphi_o + M_o H \cos(\beta - \varphi_o)]^2} = \frac{\frac{1}{2}\,\overline{K_1^2}}{[\partial^2 E_o/\partial\varphi_o^2]^2} \quad . \tag{III-45}$$

The numerator of (III-45) contains the perturbing effect of random anisotropy forces (the factor $\frac{1}{2}$ coming from the average over axis orientations α). The denominator may be thought of as a uniform magnetic stiffness tending to align $\vec{M}$ along $\vec{H}$ and along the uniform easy axis (x-axis). In strong fields ($h \gg 1$) the ripple amplitudes go to zero and the film may be considered saturated. As the field is lowered and a coherent threshold h_t is approached, $\Lambda \rightarrow 0$, the ripple amplitudes grow very large, the condition (III-3c) is violated, and we have a threshold catastrophe. The static equilibrium problem becomes nonlinear in the Fourier components $\varphi_{\vec{k}}$, and further progress is very difficult without some simplified ansatz for $\vec{M}(\vec{r})$ such as the band model of Thomas[36] or the labyrinth model of Smith and Harte.[15] Such treatments are not within the scope of this work. In the absence of the uniform stiffness, and in the absence of the interaction term $g_{\vec{k}}$, the components $\varphi_{\vec{k}}$ do not of course become infinite: $\vec{M}(\vec{r})$ just follows the random anisotropy.

A crystallite cutoff of the ripple spectrum results from the effects contained in the factor

$$(1 + r_o^2 k^2)^{-3/2} \tag{III-46}$$

of (III-39): for wavelengths $2\pi/k$ shorter than the cutoff length $2\pi r_o$, ripple amplitudes are sharply diminished. This occurs because components of the random anisotropy with wavelengths shorter than the mean crystallite size are very small.

Interaction effects, which are contained in the last factor of (III-39)

$$(1 + g_{\vec{k}})^{-2} \quad , \tag{III-47}$$

are of great importance in determining the characteristics of the magnetization ripple. The term $g_{\vec{k}}$, defined by (III-40), is the exchange plus magnetostatic interaction energy density for unit $\overline{|\varphi_{\vec{k}}|^2}$, normalized to the uniform magnetic stiffness. The effect of these interactions is to suppress all Fourier components $\varphi_{\vec{k}}$ for which $g_{\vec{k}} \gg 1$, and leave unaffected those for which $g_{\vec{k}} \ll 1$. If we define a characteristic closed curve C' in the $\vec{k}$-plane by the relation

$$g_{\vec{k}} = 1 \quad \text{for} \quad \vec{k} \text{ on } C' \tag{III-48}$$

ripple components with wave vectors on C' are attenuated by a factor of $\frac{1}{2}$ due to interactions. Since $g_{\vec{k}}$ monotonically increases with increasing k, the interactions (roughly speaking) suppress those components with wave vectors outside C' while leaving unchanged those with wave vectors inside C'.

It is helpful to transform C' to C in the reciprocal wave vector plane, defined by $2\pi\vec{k}/k^2$. Thus, points on C are described by a vector in the $\vec{k}$-direction whose length is the wavelength $2\pi/k$. Figure 5 shows the characteristic curve C for a TPF (unless otherwise noted, a TPF is

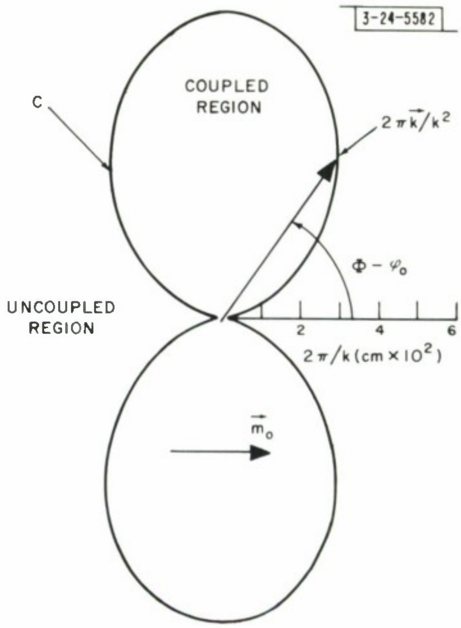

Fig. 5. Characteristic curve C showing effect of interactions on ripple components with wave vectors $\vec{k}$ (for a TPF).

in zero external field), for which $\lambda_e = 2.5 \times 10^{-5}$ cm and $\lambda_m = 1.25 \times 10^{-2}$ cm. In accord with our rough interpretation given above, all Fourier components of the ripple with wavelengths inside C are suppressed by exchange and magnetostatic interactions, while components with wavelengths outside C are unattenuated. It is easily seen that the shortest wavelength on C is for $\Phi = \varphi_o$, or wave vector in the direction of mean $\vec{M}$. The wavelength of this LMR is $2\pi\lambda_e$ (= 1.6×10^{-4} cm for a TPF), and it is apparent from the sharpness of C near the origin that this ripple is well oriented, in accord with experimental observation.

In Fig. 6 we show this region in greater detail, where now we plot angle of wave vector to mean $\vec{M}$ vs wavelength $2\pi/k$ on a logarithmic scale. Equipartition of exchange and magnetostatic energies occurs along the dashed line, which satisfies the equation

$$\lambda_e^2 k^2 = \lambda_m L^{-1} \tilde{\chi}(kL) \sin^2(\Phi - \varphi_o) \quad . \tag{III-49}$$

We see from this that over almost the whole reciprocal wave vector plane inside C the interaction is predominantly magnetostatic. For wavelengths much greater than the film thickness, or

$$kL \ll 1 \quad , \tag{III-50}$$

(III-48) becomes

$$\lambda_e^2 k^2 + \lambda_m k \sin^2(\Phi - \varphi_o) = 1 \quad . \tag{III-51}$$

The condition (III-50) is fairly well satisfied on C even for the shortest wavelength $2\pi\lambda_e$, for which $kL = L/\lambda_e$ (= 0.2 for a TPF). In the magnetostatic region we may neglect the first term in (III-51), so that C is determined by

$$(\lambda_m k)^{-1} = \sin^2(\Phi - \varphi_o) \quad . \tag{III-52}$$

From this we find that the minimum wavelength of TMR ($\vec{k} \perp$ mean $\vec{M}$) is $2\pi\lambda_m$ (= 0.8 mm for a TPF).

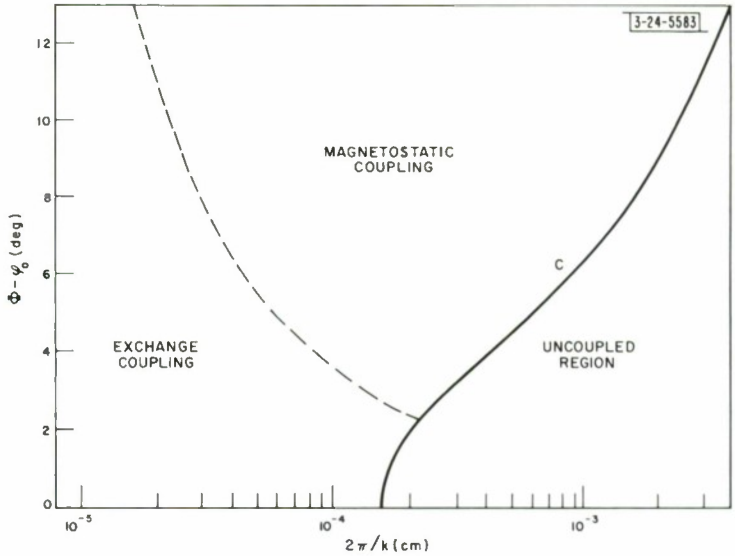

Fig. 6. Detail of Fig. 5 near origin showing exchange-coupled and magnetostatic-coupled regions (for a TPF).

The characteristic curve of Fig. 5 may also be interpreted as the extent of the interactions. $\vec{M}$ at the origin is strongly coupled to $\vec{M}$ everywhere inside C, where $2\pi\vec{k}/k^2$ has the meaning of a position vector.

We have seen that short wavelength ripple components have well-oriented wave vectors in the direction of $\vec{m}_o$ and include all wavelengths longer than about $2\pi\lambda_e$. However, electron micrographs show a fairly well-defined wavelength of 1 to 2×10^{-4} cm (see Fig. 2); one wonders why this pattern is not completely obscured by the longer-wavelength components. The answer is that what are observed are spatial variations in the intensity of the electron deflection pattern caused by spatial variations in the Lorentz force. The more rapid the variation in $\vec{M}$, the greater the electron contrast. As shown by Fuller and Hale,[28] the magnitude of the change in intensity of the electron beam due to the $\vec{k}^{th}$ component of LMR is proportional to $\sqrt{\overline{|k\varphi_{\vec{k}}|^2}}$. From (III-39),

$$\sqrt{\overline{|k\varphi_{\vec{k}}|^2}} = \frac{\Gamma}{2\Lambda} \sqrt{\pi} \frac{r_o}{\sqrt{S}} (1 + r_o^2 k^2)^{-3/4} k(1 + \lambda_e^2 k^2)^{-1} \tag{III-53}$$

so that for

$$kr_o \ll 1 \tag{III-54}$$

we find

$$\sqrt{\overline{|k\varphi_{\vec{k}}|^2}} \propto k(1 + \lambda_e^2 k^2)^{-1} \quad . \tag{III-55}$$

From (III-55) we see that the intensity spectrum has a peak at $k = \lambda_e^{-1}$. Ordinarily λ_e is much greater than the crystallite size, and (III-54) is satisfied. (For a TPF, $r_o/\lambda_e = 0.05$.) The wavelength at the peak is $2\pi\lambda_e$ ($= 1.6 \times 10^{-4}$ cm for a TPF), which corresponds quite closely to the actual wavelength observed.

With an external field applied along an easy or a hard axis of the uniform anisotropy, cases (a) and (b) of (III-38) give

$$2\pi\lambda_e = 2\pi A^{1/2}\,[K_o(h \pm 1)]^{-1/2} \quad \text{for} \quad h > \mp 1 \quad . \tag{III-56}$$

As the field is increased, the peak wavelength decreases until finally (III-54) is violated. However, this happens at a field

$$H \simeq AM_o^{-1} r_o^{-2} \quad (= 800\,\text{oe for a TPF})$$

which is so high that $\vec{M}$ is nearly saturated and the ripple has disappeared. Therefore, it is only for a film with very large crystallites ($\gtrsim 10^{-5}$ to 10^{-4} cm) that crystallite cutoff from the term

$$(1 + r_o^2 k^2)^{-3/4}$$

of (III-53) need be considered. As the field approaches a coherent threshold ($h = \mp 1$), λ_e becomes infinite — another aspect of the threshold catastrophe just described. The actual conversion of LMR into uniform magnetization is prevented by nonlinear interactions which limit the growth of the peak ripple wavelength. Similar considerations apply to case (c) of (III-38), for which

$$2\pi\lambda_e = 2\pi A^{1/2}\,[K_o(1 - h^2)]^{-1/2} \quad \text{for} \quad |h| < 1 \quad . \tag{III-57}$$

E. MAGNETIZATION DISPERSION

A vital aspect of magnetization ripple remains to be examined, namely, its over-all magnitude. Perhaps the best measure of this is the magnetization dispersion δ, the rms angular deviation of $\vec{M}(\vec{r})$ from $\vec{m}_o$, given by (III-42). This quantity is of great interest for a number of reasons. First, δ must be no smaller than the observed LMR if our theory is to account for ripple. Second, if our linearized treatment is to be valid and (III-3c) is to hold, we must find

$$\delta \ll 1 \quad . \tag{III-58}$$

Third, δ may be either measured directly or compared with "anisotropy dispersion," measured by a variety of techniques[16,17] but always near a threshold for coherent rotation. Finally, in Sec. VI we will relate dynamic effects of the ripple to δ.

From (III-43) and (III-44)

$$\delta^2 = \frac{S}{(2\pi)^2} \int_o^\infty k\,dk \int_o^{2\pi} \overline{|\varphi(\vec{k})|^2}\, d\Phi \tag{III-59}$$

where $\overline{|\varphi(\vec{k})|^2}$ is given by the right side of (III-39). Replacement of the sum over discrete $\vec{k}$ by an integral is clearly valid here, since there is no divergent contribution to the integral near $\vec{k} = 0$; for the same reason, the fact that the integral goes over $\vec{k} = 0$ while the sum did not is of no importance whatsoever in the limit $S \to \infty$.

From (III-39), and substituting $g_{\vec{k}}$ from (III-40), the integration over Φ is readily performed giving

$$\delta^2 = \frac{\Gamma^2 r_o^2}{16\Lambda^2} \int_o^\infty [(1 + r_o^2 k^2)(1 + \lambda_e^2 k^2)(1 + \lambda_e^2 k^2 + \lambda_m L^{-1}\tilde{\chi})]^{-3/2}$$
$$\cdot\,(2 + 2\lambda_e^2 k^2 + \lambda_m L^{-1}\tilde{\chi})\,k\,dk \quad . \tag{III-60}$$

We now assume that the main contribution to (III-60) comes from $k = O(\lambda_e^{-1})$. If

$$L << \lambda_e \quad , \tag{III-61}$$

which is only moderately well satisfied for a TPF ($L/\lambda_e = 0.2$), we may approximate $\tilde{\chi}(kL)$ by its first term, kL. [See Eq. (III-20).] Therefore, since for nearly all films of practical interest

$$r_o << \lambda_e << \lambda_m \tag{III-62}$$

(for a TPF, $r_o/\lambda_e = 5 \times 10^{-2}$ and $\lambda_e/\lambda_m = 2 \times 10^{-3}$), the integral in (III-60) has the approximate value (which is also a lower bound since $\tilde{\chi} < kL$)

$$\lambda_m^{-1/2} \int_0^\infty (1 + \lambda_e^2 k^2)^{-3/2} k^{1/2} \, dk \quad . \tag{III-63}$$

The integrand in (III-63) has its maximum value at $k = 1/(\lambda_e \sqrt{5})$; this gives us some confidence that the approximations used in obtaining (III-63) are good. A more rigorous approach – one which is, in fact, necessary if (III-61) or (III-62) are violated, is to split the integral of (III-60) into segments and in each segment expand each factor of the integrand about its largest term; the segments are chosen such that these expansions converge as rapidly as possible. However, this is a tedious procedure, and one which must be adapted to the relative values of the three parameters involved, namely, r_o/λ_e, L/λ_e, and λ_e/λ_m. Therefore, in the present work we shall be satisifed with the approximation (III-63).

Integrals such as (III-63), of the form

$$I(p, q) = \int_0^\infty (1 + v^2)^{-q} v^p \, dv \qquad (-1 < p < 2q - 1) \quad , \tag{III-64}$$

are easily evaluated in terms of gamma functions. Substituting $v = \tan\varphi$, (III-64) becomes

$$\begin{aligned} I(p, q) &= \int_0^{\pi/2} (\sin\varphi)^p (\cos\varphi)^{2q-p-2} \, d\varphi \\ &= \frac{1}{2} B\left(\frac{p+1}{2}, q - \frac{p+1}{2}\right) \end{aligned}$$

where $B(x, y)$ is the beta function[37] defined by

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \quad .$$

Therefore,

$$I(p, q) = \frac{\frac{1}{2} \Gamma\left(\frac{p+1}{2}\right) \Gamma\left(q - \frac{p+1}{2}\right)}{\Gamma(q)} \quad . \tag{III-65}$$

The integral in (III-63) is then

$$\frac{\frac{1}{2} \lambda_e^{-3/2} [\Gamma(\frac{3}{4})]^2}{\Gamma(\frac{3}{2})}$$

and inserting this result in (III-60) we find

$$\delta^2 = \frac{\Gamma^2(\frac{3}{4})}{16\sqrt{\pi}} \left(\frac{\Gamma r_o}{\Lambda}\right)^2 \lambda_m^{-1/2} \lambda_e^{-3/2} \quad . \tag{III-66}$$

[The gamma function, always written with an argument, is not to be confused with Γ, defined by (III-27).]

Using (III-36) and (III-37), and evaluating the numerical constant, we have the final result for the dispersion

$$\boxed{\delta = 0.145 K_1 r_o M_o^{-1/2} L^{-1/4} (A K_o \Lambda)^{-3/8}} \tag{III-67}$$

where by K_1 we mean the rms value $\sqrt{\overline{K_1^2}}$.

For a TPF

$$\delta = 0.076 \Lambda^{-3/8} \tag{III-68}$$

on an rms ripple of about 4° in zero field. This is certainly a reasonable order of magnitude, although it must be remembered that the value chosen for K_1 (5×10^4 erg/cm^3) is a very rough estimate. It should be kept in mind that this dispersion includes not merely LMR, but components with wave vectors in all directions and of all wavelengths. Therefore, we expect it to be somewhat larger than LMR as measured by Lorentz microscopy.

For an isotropic film ($K_o \to 0$), $K_o \Lambda \to \frac{1}{2} M_o H$ in (III-67); as the external field $H \to 0$, the dispersion grows very large until (III-58) is violated. This is a long wavelength divergence (not unlike the divergence of spontaneous magnetization in a two-dimensional lattice[38]). The problem here is that the Fourier components become strongly coupled (see Appendix B), and the iteration procedure which led to (III-39) breaks down. In fact, for an isotropic film in zero field our Fourier transform method is not a good one, and an approach such as that used by Rother[33] is better. (His main result is a ripple with mean wavelength proportional to r_o and magnitude proportional to $K_1 r_o^2/A$, where the constants of proportionality are ~10.) Using another method, which differs considerably from ours, Hoffmann[39,40] has recently obtained the ripple wavelength given by (III-56) and amplitude given by (III-67) with a slightly different numerical factor.

Summarizing the results of Sec. III-E, we find the magnetization dispersion: (a) to be typically a few degrees; (b) to be proportional to K_1 and the crystallite size; (c) to vary as the $-\frac{3}{8}$ power of the uniform anisotropy and of $h \pm 1$ (for fields along $\vec{m}_o$ in the easy and hard directions, respectively); and (d) to depend only weakly on the film thickness.

IV. UNIFORM MAGNETIZATION REVERSAL

A. INTRODUCTION

Before we consider the dynamic behavior of the magnetization ripple, it is pertinent to examine the process of uniform magnetization reversal, i.e., coherent rotation with $\vec{M}(\vec{r}, t) = \vec{m}_o(t)$. This process is the starting point in the treatment of nonuniform reversal to follow, and it is important to understand how it occurs. Furthermore, earlier treatments[2,8,41] of rotational switching have been based on the uniform rotation model (but with a phenomenological damping term included in place of detailed knowledge of the dissipative processes involved). After a short description of the uniform rotational mode, we will formulate the equations of motion in an anisotropic film; in Sec. V they will be used in obtaining the dynamic response of the ripple (no explicit solution will be necessary for this purpose). Then we will solve these equations, with and without a damping term, for an isotropic film; the overdamped case will be treated somewhat more rigorously than has been done previously.

Rotational switching in a film is a two-step process. When a pulse magnetic field $\vec{H}_p$ (typically a few oersteds) is applied in the film plane at time $t = 0$, the magnetization $\vec{m}_o(t)$, initially lying in the plane, starts to precess about $\vec{H}_p$. In doing so it lifts slightly out of the plane (typically 10^{-2} radian), creating a normal demagnetizing field equal and opposite to the normal component of $4\pi\vec{m}_o$ (typically 10^2 oe). The magnetization now processes about this larger field in a nearly planar path.

However, this uniform mode has acquired an energy $-\vec{H}_p \cdot \vec{m}_o(0)$ from the external field; as a result, in the absence of any dissipative mechanisms, $\vec{m}_o$ must continue to precess indefinitely. If there is a slow energy loss, $\vec{m}_o$ will experience damped oscillations about its equilibrium direction near $\vec{H}_p$ (at $\vec{H}_p$ for an isotropic film); if the energy loss is rapid, $\vec{m}_o$ is overdamped and reaches equilibrium without oscillations. In many switching experiments oscillations are not observed (but see, for example, Dietrich and Proebster[42] and Hearn[10]), so that in the phenomenological theory of Sec. IV-D the overdamped case is of great practical interest.

B. EQUATIONS OF MOTION

The gyromagnetic equation for a uniform magnetic moment density $\vec{m}_o(t)$ is [see (I-2)]

$$\frac{d\vec{m}_o}{dt} = -\gamma\vec{m}_o \times \vec{h}_o + \text{damping term} \tag{IV-1}$$

where γ is the absolute value of the gyromagnetic ratio and $\vec{h}_o$ is the spatial average of the effective magnetic field in the ferromagnet. For a thin film this effective field may be taken as the external field (pulse and steady), the uniaxial anisotropy field, and the uniform demagnetizing field $-4\pi\vec{m}_o \cdot \vec{i}_z\vec{i}_z$ [see (II-24a)].

As long as we remain ignorant of the specific loss mechanisms, we can only guess at the form of the damping term. Landau and Lifshitz[12] proposed

$$-\frac{\lambda}{|\vec{m}_o|^2}\,\vec{m}_o \times (\vec{m}_o \times \vec{h}_o) \tag{IV-2}$$

where λ is a positive constant of dimensions $\sec^{-1}$. A slight modification was suggested by Gilbert,[18] who used a Lagrangian approach and a Rayleigh dissipation function to arrive at

$$\frac{d\vec{m}_o}{dt} = -\gamma \vec{m}_o \times \vec{h}_o + \frac{\alpha}{|\vec{m}_o|} \vec{m}_o \times \frac{d\vec{m}_o}{dt} \tag{IV-3}$$

where α is a dimensionless positive constant. However, it is easy to show that these damping terms are equivalent, except for a contraction of the time scale. Taking the dot product of $\vec{m}_o$ with (IV-3), we see that

$$\vec{m}_o \cdot \frac{d\vec{m}_o}{dt} = 0 \tag{IV-4a}$$

or

$$|\vec{m}_o| = \text{const.} \equiv M_o \quad . \tag{IV-4b}$$

Note that $|\vec{m}_o|$ is also conserved if a damping term (IV-2) is assumed instead. Now, taking the cross product of $\vec{m}_o$ with (IV-3), using (IV-4a, b), and rearranging terms, we find

$$(1 + \alpha^2) \frac{d\vec{m}_o}{dt} = -\gamma \vec{m}_o \times \vec{h}_o - \frac{\alpha\gamma}{M_o} \vec{m}_o \times (\vec{m}_o \times \vec{h}_o) \quad . \tag{IV-5}$$

Thus, if we let $t' = (1 + \alpha^2)^{-1} t$ in (IV-5) and make the substitution

$$\lambda = \alpha\gamma M_o \tag{IV-6}$$

the Landau-Lifshitz and Gilbert forms are identical. The contraction of the time scale is unimportant if $\alpha << 1$, but as $\alpha \to \infty$, (IV-5) gives

$$\frac{d\vec{m}_o}{dt} \to 0 \tag{IV-7a}$$

whereas

$$\frac{d\vec{m}_o}{dt'} \to \infty \quad . \tag{IV-7b}$$

In other words, in the limit of large damping, the switching speed goes to zero for Gilbert damping but becomes infinite for Landau-Lifshitz damping. The latter situation seems physically unreasonable, as pointed out by Kikuchi.[41] Callen[43] has suggested a more general form of the damped gyromagnetic equation, the various parameters of which may be associated with specific physical processes. However, the difficulties one would encounter in attempting to use this approach to describe a switching process (in which $\vec{m}_o$ is initially far from equilibrium) are probably insurmountable. The Gilbert equation will suffice for the purpose of this section, and we now proceed from (IV-5).

We shall use a spherical coordinate system for $\vec{m}_o$ as shown in Fig. 7 where the + x-direction is the easy direction of uniaxial anisotropy nearest $\vec{m}_o(0)$. In an isotropic film we take $\vec{m}_o(0) = M_o \vec{i}_x$. If we let $E_o(\theta_o, \varphi_o)$ be the energy of $\vec{m}_o$ in the external, demagnetizing, and anisotropy fields, then the effective field is

$$\vec{h}_o = -\nabla_{\vec{m}_o} E_o \tag{IV-8}$$

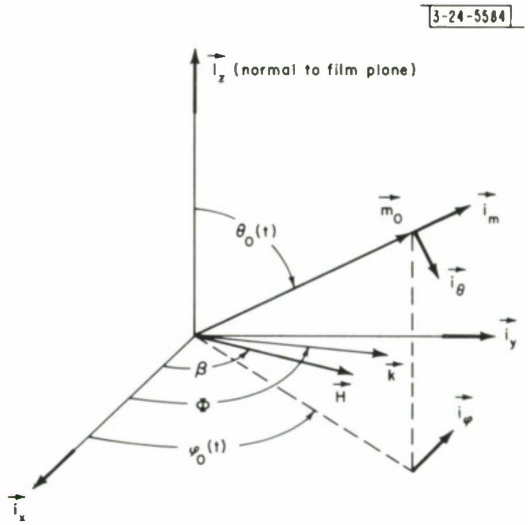

Fig. 7. Coordinate system for dynamic case.

and the spatial components of (IV-5) are

$$(1+\alpha^2)\frac{d\Theta_o}{dt} = -\frac{\gamma}{M_o}\frac{\partial E_o}{\partial\varphi_o} - \frac{\alpha\gamma}{M_o}\frac{\partial E_o}{\partial\Theta_o} \tag{IV-9a}$$

and

$$(1+\alpha^2)\sin\Theta_o\frac{d\varphi_o}{dt} = \frac{\gamma}{M_o}\frac{\partial E_o}{\partial\Theta_o} - \frac{\alpha\gamma}{M_o}\frac{\partial E_o}{\partial\varphi_o} \quad . \tag{IV-9b}$$

We consider an external field $\vec{H}$ in the film plane at an angle β to the x-axis. Then the energy E_o is given by

$$E_o = -\vec{m}_o\cdot\vec{H} + 2\pi(\vec{m}_o\cdot\vec{i}_z)^2 - K_oM_o^{-2}(\vec{m}_o\cdot\vec{i}_x)^2$$

$$= -M_oH\sin\Theta_o\cos(\varphi_o-\beta) + 2\pi M_o^2\cos^2\Theta_o - K_o\sin^2\Theta_o\cos^2\varphi_o \tag{IV-10}$$

with derivatives

$$\frac{1}{M_o}\frac{\partial E_o}{\partial\Theta_o} = -H\cos\Theta_o\cos(\varphi_o-\beta) - 2\pi M_o\sin 2\Theta_o - \frac{1}{2}H_K\sin 2\Theta_o\cos^2\varphi_o \tag{IV-11a}$$

$$\frac{1}{M_o}\frac{\partial E_o}{\partial\varphi_o} = H\sin\Theta_o\sin(\varphi_o-\beta) + \frac{1}{2}H_K\sin^2\Theta_o\sin 2\varphi_o \tag{IV-11b}$$

where the anisotropy field is defined by

$$H_K = \frac{2K_o}{M_o} \quad . \tag{IV-12}$$

We next let

$$\psi_o = \Theta_o - \frac{\pi}{2} \tag{IV-13}$$

and make the key assumption that during the entire switching process $\vec{m}_o$ is confined nearly to the film plane, or

$$|\psi_o| << 1 \quad . \tag{IV-14}$$

It is easy to see from the energy expression (IV-10) that the condition (IV-14) will be well satisfied provided

$$4\pi M_o >> H \tag{IV-15}$$

because the magnetostatic energy created in lifting $\vec{m}_o$ out of the film plane an appreciable angle is then much greater than the energy available from the external field. The requirement (IV-15) is indeed met for all experimental situations of current interest. Also, we shall verify (IV-14) for the solutions to be found in Secs. IV-C and IV-D. The condition (IV-14) will enable us to separate the coordinates in (IV-9a, b), but even if it were violated we could still proceed in the next section with the calculation of the dynamic response of magnetization ripple. However, the simplification introduced by (IV-14) is so great that at the risk of some loss in generality we will assume it holds in all that follows. Then with the further requirement

$$4\pi M_o >> H_K \tag{IV-16}$$

which is invariably met in practice, (IV-11a, b) become

$$\frac{1}{M_o}\frac{\partial E_o}{\partial \theta_o} \simeq 4\pi M_o \psi_o \tag{IV-17a}$$

$$\frac{1}{M_o}\frac{\partial E_o}{\partial \varphi_o} \simeq H \sin(\varphi_o - \beta) + \frac{1}{2} H_K \sin 2\varphi_o \quad . \tag{IV-17b}$$

We next generalize (IV-17b). Consider a step function pulse field

$$\begin{aligned} \vec{H}_p(t) &= \vec{H}_p = \text{const.} \qquad t > 0 \\ &= 0 \qquad\qquad\qquad t < 0 \end{aligned} \tag{IV-18}$$

at an angle β_p to the x-axis, and a constant bias field $\vec{H}_b$ at an angle β_b. Then (IV-17b) becomes

$$\frac{1}{M_o}\frac{\partial E_o}{\partial \varphi_o} = H_p \sin(\varphi_o - \beta_p) + H_b \sin(\varphi_o - \beta_b) + \frac{1}{2} H_K \sin 2\varphi_o \tag{IV-17b-1}$$

for $t > 0$, and (IV-17a) is of course unchanged. We note at this point that in the usual experimental arrangements either

$$\beta_p = \pi \quad , \quad \beta_b = \frac{\pi}{2} \tag{IV-19a}$$

or

$$0 < \beta_p < \pi \quad , \quad H_b = 0 \quad . \tag{IV-19b}$$

In the first case [(IV-19a)], that of a hard axis bias field and an easy axis switching field, the notation

$$\begin{aligned} H_p &\rightarrow H_{||} \\ H_b &\rightarrow H_{\perp} \end{aligned} \tag{IV-19c}$$

is conventional. In the second case (IV-19b), 90° switching with $\beta_p = \pi/2$ is most common. Other arrangements are possible, but experimental difficulties invariably render them undesirable.

Inserting (IV-17a) and (IV-17b-1) in (IV-9a, b), and using (IV-13) and (IV-14), we find

$$(1 + \alpha^2) \frac{d\psi_o}{dt} = \omega_h \rho(\varphi_o) - \alpha\omega_m \psi_o \tag{IV-20a}$$

$$(1 + \alpha^2) \frac{d\varphi_o}{dt} = \omega_m \psi_o + \alpha\omega_h \rho(\varphi_o) \tag{IV-20b}$$

where

$$\omega_h = \gamma H_p \tag{IV-21a}$$

$$\omega_m = \gamma 4\pi M_o \tag{IV-21b}$$

$$-\rho(\varphi_o) = \sin(\varphi_o - \beta_p) + \frac{H_b}{H_p} \sin(\varphi_o - \beta_p) + \frac{1}{2} \frac{H_K}{H_p} \sin 2\varphi_o \quad . \tag{IV-21c}$$

The precessional frequency in the pulse field is ω_h and is typically $5 \times 10^7 \sec^{-1}$; ω_m is the precessional frequency in the demagnetizing field for $\vec{m}_o$ perpendicular to the film plane and is $1.9 \times 10^{11} \sec^{-1}$ for a TPF; $\rho(\varphi_o)$ is proportional to the planar torque on $\vec{m}_o$. It is convenient to introduce a dimensionless time variable

$$\tau = \sqrt{\omega_h \omega_m}\, t \tag{IV-22}$$

and to normalize the angular deviation of $\vec{m}_o$ from the film plane by the substitution

$$\sigma_o = \epsilon^{-1} \psi_o \tag{IV-23}$$

where

$$\epsilon = \sqrt{\frac{\omega_h}{\omega_m}} << 1 \quad . \tag{IV-24}$$

The small parameter ϵ is typically 1.5×10^{-2}. We also define a normalized damping parameter

$$\nu = \frac{\alpha}{\epsilon} \quad . \tag{IV-25}$$

Equations (IV-20a, b) then become

$$(1 + \epsilon^2\nu^2)\, \dot{\sigma}_o = \rho(\varphi_o) - \nu\sigma_o \tag{IV-26a}$$

$$(1 + \epsilon^2\nu^2)\, \dot{\varphi}_o = \sigma_o + \epsilon^2 \nu\rho(\varphi_o) \tag{IV-26b}$$

where the dot indicates differentiation with respect to τ. For the purposes of Sec. V, the phenomenological damping will be small ($\nu < 1$), and (IV-26a, b) may be approximated by

$$\boxed{\begin{aligned} \dot{\sigma}_o &= \rho(\varphi_o) - \nu\sigma_o \\ \dot{\varphi}_o &= \sigma_o \end{aligned}} \quad . \qquad \begin{aligned} &(\text{IV-27a}) \\ &(\text{IV-27b}) \end{aligned}$$

The qualitative description of rotational switching given in Sec. IV-A is reflected in (IV-26a, b) [or (IV-27a, b)]. The primary precession about the external field (plus uniaxial anisotropy field) is described by (IV-26a), and the secondary precession about the normal demagnetizing field by (IV-26b). We see that it is mainly the primary precession which is damped; this leads to a reduced demagnetizing field and therefore a slower secondary precession.

Initial values of σ_o and φ_o are determined by minimizing the energy E_o for $t < 0$. Equations (IV-17a) and (IV-17b-1) give [using (IV-23)]

$$\sigma_o(0) = 0 \tag{IV-28a}$$

$$H_b \sin[\varphi_o(0) - \beta_b] + \frac{1}{2} H_K \sin 2\varphi_o(0) = 0 \tag{IV-28b}$$

with the added condition

$$\left.\frac{1}{M_o} \frac{\partial^2 E_o}{\partial \varphi_o^2}\right|_{t<0} = H_b \cos[\varphi_o(0) - \beta_b] + H_K \cos 2\varphi_o(0) > 0 \quad . \tag{IV-29}$$

For the usual arrangement $\beta_b = \pi/2$, (IV-28b) and (IV-29) give

$$\varphi_o(0) = \arcsin \frac{H_b}{H_K} \qquad (0 \leqslant H_b < H_K) \tag{IV-30a}$$

$$= \frac{\pi}{2} \qquad (H_b \geqslant H_K) \quad . \tag{IV-30b}$$

[The case (IV-30b) is trivial, as no switching occurs.]

To conclude Sec. IV-B, we examine the behavior of the uniform mode in the vicinity of its new equilibrium direction at the completion of the switching process. This direction is determined by

$$\lim_{t\to\infty} \frac{d\vec{m}_o}{dt} = 0 \tag{IV-31}$$

or, equivalently, that the energy E_o is minimized for σ_o and φ_o at $t \to \infty$. Either way, we find

$$\lim_{t\to\infty} \sigma_o(t) \equiv \check{\sigma}_o = 0 \tag{IV-32a}$$

$$\lim_{t\to\infty} \rho[\varphi_o(t)] \equiv \check{\rho}(\check{\varphi}_o) = 0 \quad . \tag{IV-32b}$$

We expand (IV-26a, b) about $\check{\sigma}_o$ and $\check{\varphi}_o$ and, assuming that σ_o and $(\varphi_o - \check{\varphi}_o)$ vary as $\exp[\Omega(1 + \epsilon^2\nu^2)^{-1}\tau]$, arrive at the characteristic determinant

$$\begin{vmatrix} \Omega + \nu & -\check{\rho}' \\ -1 & \Omega - \epsilon^2\nu\check{\rho}' \end{vmatrix} = 0 \tag{IV-33}$$

where

$$\check{\rho}' = \left.\frac{d\rho}{d\varphi_o}\right|_{\varphi_o = \check{\varphi}_o} \quad . \tag{IV-34}$$

Dropping terms of order ϵ^2, we extract from (IV-33) the critical damping

$$\nu_c = 2\sqrt{-\check{\rho}'} \qquad \text{(IV-35)}$$

or, with the help of (IV-21c) and (IV-24),

$$\alpha_c = \frac{2}{\sqrt{4\pi M_o}} [H_p \cos(\check{\varphi}_o - \beta_p) + H_b \cos(\check{\varphi}_o - \beta_b) + H_K \cos 2\check{\varphi}_o]^{1/2} \quad . \qquad \text{(IV-36)}$$

If $\alpha < \alpha_c$, $\vec{m}_o$ overshoots equilibrium and experiences a damped oscillation; if $\alpha > \alpha_c$, $\vec{m}_o$ reaches equilibrium without oscillations. Note that if $\check{\rho}' > 0$, the equilibrium direction $\check{\varphi}_o$ is unstable; thus the threshold field for irreversible rotation through any equilibrium point $\check{\varphi}_o$ is determined by

$$\check{\rho} = \check{\rho}' = 0 \quad . \qquad \text{(IV-37)}$$

(See p. 22.)

C. UNDAMPED UNIFORM MODE

An independent measurement of α may be obtained from the linewidth in ferromagnetic resonance. Although there may be dissipative processes in large-angle switching which are not operative in small-angle resonance (and indeed it is one purpose of this work to uncover such processes), the microwave resonance damping constant α_{res} provides a measure of "intrinsic" loss mechanisms (the details of which we will make no attempt to treat) and a lower bound on the total damping. Various measurements in Permalloy films at microwave frequencies[10,14,15] give $\alpha_{res} \simeq 10^{-2}$, which is barely less than α_c for the usual switching experiments.

As an approximation to this slightly underdamped situation, we solve (IV-27a, b) with $\nu = 0$ for an isotropic film without a bias field. The equations of motion, from (IV-21c) and (IV-27a, b), are

$$\dot{\sigma}_o = -\sin(\varphi_o - \beta) \qquad \text{(IV-38a)}$$

$$\dot{\varphi}_o = \sigma_o \qquad \text{(IV-38b)}$$

with initial conditions

$$\sigma_o(0) = \varphi_o(0) = 0 \quad . \qquad \text{(IV-38c)}$$

Eliminating σ_o from (IV-38a, b), we obtain

$$\ddot{\varphi}_o + \sin(\varphi_o - \beta) = 0 \qquad \text{(IV-39)}$$

with initial conditions

$$\varphi_o(0) = \dot{\varphi}_o(0) = 0 \quad . \qquad \text{(IV-40)}$$

Equation (IV-39) is the equation of a simple pendulum, and is readily solved by Jacobian elliptic functions.[44] We introduce a new dependent variable

$$u = b \tan \frac{1}{2} (\beta - \varphi_o) \qquad \text{(IV-41)}$$

where

$$b = \cot \frac{\beta}{2} \quad . \qquad \text{(IV-42)}$$

Equations (IV-39) and (IV-40) then become

$$\ddot{u} - \frac{2u\dot{u}^2}{b^2 + u^2} + u = 0 \tag{IV-43a}$$

$$u(0) = 1 \tag{IV-43b}$$

$$\dot{u}(0) = 0 \quad . \tag{IV-43c}$$

An integrating factor for (IV-43a) is $(b^2 + u^2)^{-2}$, from which we find the first integral

$$\dot{u}^2 = \frac{(b^2 + u^2)(1 - u^2)}{b^2 + 1} \quad . \tag{IV-44}$$

A second integration gives us

$$u = \mathrm{cn}(\tau, \sin\frac{\beta}{2}) \tag{IV-45}$$

which is the main result of Sec. IV-C.

As expected for a conservative system, the motion is periodic, with period $4K(\sin\beta/2)$, where K is the complete elliptic integral of the first kind. The switching time τ_s may be defined as the time it takes $\vec{m}_o$ to rotate from $\varphi_o = 0$ to $\varphi_o = \beta$, the new equilibrium angle, and is

$$\tau_s = K(\sin\frac{\beta}{2}) \tag{IV-46}$$

or

$$t_s = \gamma^{-1}(4\pi M_o H)^{-1/2} K(\sin\frac{\beta}{2}) \quad . \tag{IV-47}$$

Finally, we must verify (IV-14). From (IV-38b) and (IV-41),

$$\sigma_o = \frac{-2b}{b^2 + u^2}\dot{u} \tag{IV-48}$$

and using (IV-45), we find

$$\sigma_o = \sin\beta \frac{\mathrm{sn}(\tau, \sin\frac{\beta}{2})}{\mathrm{dn}(\tau, \sin\frac{\beta}{2})} \quad . \tag{IV-49}$$

This function attains its maximum value at $\tau = \tau_s$ [see (IV-46)], and is

$$\sigma_o(\tau_s) = 2\sin\frac{\beta}{2} \quad . \tag{IV-50}$$

Thus

$$(\psi_o)_{max} = 2\epsilon \sin\frac{\beta}{2} << 1 \quad . \tag{IV-51}$$

D. OVERDAMPED UNIFORM MODE

In contrast to the damping deduced from microwave resonance linewidths, the damping actually observed in rotational switching experiments is quite high, particularly in the intermediate-speed region (see p. 3), where one invariably finds

$$\nu >> 1 \qquad \left(\alpha^2 >> \frac{H_p}{4\pi M_o}\right) \quad . \tag{IV-52}$$

We leave for Sec. VI the question of why this damping should be so great and solve (IV-26 a, b), for an isotropic film with a damping constant satisfying (IV-52), by a boundary layer method.[45]

We first eliminate σ_o from (IV-26a, b), obtaining

$$(1 + \epsilon^2\nu^2)\,\ddot{\varphi}_o + \nu\dot{\varphi}_o - \rho = 0 \tag{IV-53}$$

where for an isotropic film

$$\rho(\varphi_o) = -\sin(\varphi_o - \beta) \tag{IV-54a}$$

$$\varphi_o(0) = 0 \tag{IV-54b}$$

$$\dot{\varphi}_o(0) = \frac{\epsilon^2\nu\sin\beta}{1 + \epsilon^2\nu^2} \quad . \tag{IV-54c}$$

Next we introduce a new time variable

$$\bar{\tau} = \nu^{-1}\tau = \frac{\gamma H}{\alpha}\, t \tag{IV-55}$$

and transform (IV-53) to an equation for $u(\bar{\tau})$, where u is defined by (IV-41). The result is

$$\delta\left[\frac{d^2u}{d\bar{\tau}^2} - \frac{2u}{b^2 + u^2}\left(\frac{du}{d\bar{\tau}}\right)^2\right] + \frac{du}{d\bar{\tau}} + u = 0 \tag{IV-56a}$$

with

$$u(0) = 1 \tag{IV-56b}$$

and

$$\left.\frac{du}{d\bar{\tau}}\right|_{\bar{\tau}=0} = -\epsilon^2\delta^{-1} = \frac{-\alpha^2}{1 + \alpha^2} \tag{IV-56c}$$

where

$$\delta = \frac{1 + \epsilon^2\nu^2}{\nu^2} = \frac{\epsilon^2}{\alpha^2}(1 + \alpha^2) << 1 \quad . \tag{IV-57}$$

We start our analysis with (IV-56a-c). It might seem at first glance that since δ is a small parameter, we could neglect the first term of (IV-56a), retaining the second (damping) and third (restoring torque) terms. This is the "viscous flow" approximation of Smith[4] (which reduces the problem to an integration for the anisotropic case). However, this solution is not valid in the neighborhood of $\bar{\tau} = 0$ because there (unless $\alpha > 1$) $du/d\bar{\tau}$ is small and the first term in (IV-56a) is of order unity. This boundary layer at $\bar{\tau} = 0$ must be accounted for properly in order to apply the initial conditions (IV-56b) and (IV-56c) to the "viscous flow" solution valid past the boundary layer.

Let the solution beyond the (as yet unspecified) boundary layer be $\bar{u}(\bar{\tau})$, which we expand as a power series in δ:

$$\bar{u}(\bar{\tau}) = \sum_{i=0}^{\infty} \bar{u}_i(\bar{\tau})\, \delta^i \quad . \tag{IV-58}$$

Inserting (IV-58) in (IV-56a), we find that the solution to zeroth order in δ is

$$\bar{u}_o = C_o\, e^{-\bar{\tau}} \tag{IV-59}$$

where C_o is a constant which must be determined. Higher order terms $\bar{u}_1, \bar{u}_2, \ldots$ can be found without difficulty, but we shall not give them explicitly. It is obvious from (IV-56a) that they will each contain a single undetermined constant $C_1, C_2, \ldots$.

To solve (IV-56a-c) within the boundary layer, we first expand the time scale through the change of variable

$$v = \delta^m \bar{\tau} \qquad (m < 0) \tag{IV-60}$$

where m will be chosen such that the boundary layer, in which the first term of (IV-56a) is not negligible, is given by $0 \leqslant v \lesssim 1$. We next let the complete solution to (IV-56a-c), valid for all time, be

$$u = \bar{u}(\bar{\tau}) + \delta^n w(v) \qquad (n > 0) \tag{IV-61}$$

where n will be chosen such that $w = O(1)$ for $0 \leqslant v \lesssim 1$. Since $u \rightarrow \bar{u}$ for $v >> 1$, we require

$$\lim_{v \rightarrow \infty} w(v) = 0 \quad . \tag{IV-62}$$

With these new variables, (IV-56a-c) become

$$\delta\bar{u}'' + \delta^{2m+n+1} w'' - 2\delta\, \frac{(\bar{u} + \delta^n w)\,(\bar{u}' + \delta^{m+n} w')^2}{b^2 + (\bar{u} + \delta^n w)^2} + \bar{u}' + \delta^{m+n} w' + \bar{u} + \delta^n w = 0 \tag{IV-63a}$$

$$u(0) = \bar{u}(0) + \delta^n w(0) = 1 \tag{IV-63b}$$

$$u'(0) = \bar{u}'(0) + \delta^{m+n} w'(0) = -\delta^{-1}\epsilon^2 = \frac{-\alpha^2}{1 + \alpha^2} > -1 \tag{IV-63c}$$

where $\bar{u}' \equiv d\bar{u}/d\bar{\tau}$, and $w' \equiv dw/dv$. Now let us assume that w is a well-behaved function so that w' and w'' are of order unity for $0 \leqslant v \lesssim 1$. Then the boundary-layer width is determined by the condition that coefficients of w' and w'' in (IV-63a) are of equal order in δ, giving

$$m = -1 \quad . \tag{IV-64}$$

The initial values of w and w' are found from (IV-58), (IV-61), (IV-63b), and (IV-63c) to be

$$w(0) = \delta^{-n}\, [1 - \bar{u}_o(0) - \delta\bar{u}_1(0) - \ldots] \tag{IV-65a}$$

and

$$w'(0) = \delta^{1-n}\, [-\delta^{-1}\epsilon^2 - \bar{u}_o'(0) - \delta\bar{u}_1'(0) - \ldots] \quad . \tag{IV-65b}$$

The exponent n and the constant C_o of (IV-59) are then determined through the requirement that $w(0)$ and $w'(0)$ remain of order unity as $\delta \rightarrow 0$. From the first two terms in (IV-65a) we must have either $n = 0$ or $\bar{u}_o(0) = 1$. However, if $n = 0$, the leading terms in $w'(0)$ are of order δ and ϵ^2, in violation of the above requirement. Therefore $\bar{u}_o(0) = 1$, or from (IV-59),

$$C_o = 1 \quad . \tag{IV-66}$$

But now $\bar{u}_o'(0) = -1$, so that the largest term in (IV-65b) is δ^{1-n} (remembering that $\delta^{-1}\epsilon^2 < 1$), from which we see that

$$n = 1 \quad . \tag{IV-67}$$

Next we expand w as a power series in δ:

$$w(v) = \sum_{i=0}^{\infty} w_i(v)\, \delta^i \quad . \tag{IV-68}$$

Returning to (IV-63a) with $m = -1$ and $n = 1$, we find to zeroth order in δ

$$w_o'' + w_o' = -\bar{u}_o' - \bar{u}_o = 0 \tag{IV-69a}$$

with initial conditions given by (IV-65a, b):

$$w_o(0) = -\bar{u}_1(0) \tag{IV-69b}$$

$$w_o'(0) = -\bar{u}_o'(0) - \frac{\alpha^2}{1+\alpha^2} = \frac{1}{1+\alpha^2} \quad . \tag{IV-69c}$$

The solution to (IV-69a) is

$$w_o = D_o\, e^{-v} + \bar{D}_o \tag{IV-70}$$

where D_o and $\bar{D}_o$ are constants determined by (IV-69c) and (IV-62), respectively. The result is

$$w_o = -(1+\alpha^2)^{-1}\, e^{-v} \quad . \tag{IV-71}$$

The condition (IV-69b) determines $\bar{u}_1(0)$, i.e., the constant C_1. Then from the next term in (IV-68), w_1, comes the constant C_2 in $\bar{u}_2$, and so on to any desired order.

Collecting our results, we have found

$$u = e^{-\bar{\tau}} + O(\delta) \tag{IV-72a}$$

and

$$\frac{du}{d\bar{\tau}} = -e^{-\bar{\tau}} + \frac{1}{1+\alpha^2}\, e^{-\bar{\tau}/\delta} + O(\delta) \tag{IV-72b}$$

where the second term in (IV-72b) is of importance only within the boundary layer defined by

$$0 \leqslant \bar{\tau} \lesssim \delta << 1 \tag{IV-73a}$$

or in real time by

$$0 \leqslant t \lesssim \frac{1+\alpha^2}{\gamma 4\pi M_o \alpha} \quad . \tag{IV-73b}$$

The switching time for this overdamped case may be defined conveniently as the decay time of u, which means

$$\bar{\tau}_s = 1 \tag{IV-74a}$$

or in real time

$$t_s = \frac{\alpha}{\gamma H} \quad . \tag{IV-74b}$$

From (IV-26b), (IV-41), (IV-55), and (IV-72a, b)

$$\sigma_o = \frac{1}{\nu} \frac{2b}{b^2 + e^{-2\bar{\tau}}} (e^{-\bar{\tau}} - e^{-\bar{\tau}/\delta}) \quad . \tag{IV-75}$$

Thus from (IV-23) and (IV-42),

$$\begin{aligned} (\psi_o)_{max} &\simeq \frac{\epsilon}{\nu} && \text{for} \quad \pi > \beta > \frac{\pi}{2} \\ &\simeq \frac{\epsilon}{\nu} \sin\beta && \text{for} \quad \frac{\pi}{2} \geqslant \beta > 0 \end{aligned} \tag{IV-76}$$

and (IV-14) is satisfied. A simple physical interpretation can be found for the boundary layer: it is the initial interval during which the growth of the component of $\vec{m}_o$ normal to the film plane takes place, i.e., the time for the primary precession. [From (IV-73b) we see that this time has a minimum at $\alpha = 1$ of $2/\omega_m \sim 10^{-11}$ sec.]

The qualitative results we have obtained here for overdamped switching (and in Sec. IV-C for undamped switching) will still be valid for anisotropic films. But if accurate numerical results are desired, the presence of anisotropy (except for some limiting cases) so complicates the integrations that a computer solution becomes advisable. However, a rather good approximation is to replace H in the solutions for φ_o and σ_o by $H_p - H_{pt}$, where H_{pt} is the threshold field for irreversible rotation [see (IV-37)].

V. SPIN-WAVE THEORY: QUASISTATIC SPECTRUM AND TRANSIENT RESPONSE

A. INTRODUCTION

We now turn to the dynamic behavior of magnetization fluctuations, i.e., spin waves, paying particular attention to the transient response of ripple to a pulse field; we also obtain the quasi-static spin-wave spectrum and briefly discuss instabilities. The microwave resonance situation, in which the amplitudes of uniform and nonuniform modes are small, is rather well understood in the short-wavelength limit[43,46] (in which magnetostatic fields for all modes with $\vec{k} \neq 0$ may be replaced by their infinite medium values) and in the long-wavelength, or magnetostatic limit[47,48] (in which exchange fields may be neglected). Our treatment differs from these in two important respects. First, the amplitude of the uniform mode cannot be considered small (it may be 180°) and must therefore be retained to all orders. We avoid some of the difficulty this nonlinearity introduces by using a coordinate system rotating with the uniform mode. Second, neither the infinite medium nor the magnetostatic approximations are adequate for a thin film in low fields, since the spin waves most important in switching have wavelengths somewhat greater than the film thickness and also involve exchange fields. In resonance also, these approximations break down.[49] This difficulty is surmounted with the help of the thin-film approximation (TFA) of Sec. II, which enables us to consider magnetostatic and exchange fields simultaneously. (This has already been used to obtain the static solution of Sec. III.)

We shall assume that random, local anisotropy forces have no first-order dynamic effects. Since these forces are, on the average, isotropic, their dynamic perturbing effects should be approximately as great as their static effects; but we shall find that magnetostatic fields grow very large during magnetization reversal, so that we may expect local anisotropy fields to be relatively unimportant. This, together with the assumption that fluctuations of $\vec{M}$ from $\vec{m}_o$ are small [see (I-1a)], implies that ripple components are uncoupled dynamically (as well as statically), since the only linear coupling is via spatially varying anisotropy fields, as was shown in Sec. III.

Another assumption we make, in order to avoid excessive complications, is that damping may be neglected. As far as the uniform mode is concerned, the intrinsic (resonance) damping is small, as was noted in Sec. IV; the large damping observed in switching is precisely what we are attempting to find a physical basis for, and we therefore avoid any phenomenological treatment of it. Damping of spin waves can safely be neglected if it results in relaxation times which are longer than the switching time. If, on the other hand, relaxation times are short, the initial Fourier components do not maintain their identity during switching, and the problem is quite different. Since the important components are relatively long wavelength ($\sim 10^{-4}$ cm), we may expect their damping to be about the same as that of the uniform mode in resonance, and we are, therefore, justified in neglecting it.

Finally, we assume that the z-dependence of $\delta\vec{M}(\vec{r}, t)$ may be ignored (z-axis perpendicular to the film plane). We saw in Sec. III that exchange fields sharply attenuate all components with wavelengths shorter than an exchange wavelength $2\pi\lambda_e$ [see (III-36)] which is typically an order of magnitude greater than the film thickness. Thus, any components with appreciable z-dependence would be expected to have very small amplitudes. In somewhat thicker films ($\gtrsim 2000$ Å), standing, z-directed spin waves,[50] which can be excited by microwave fields and which

depend on the boundary conditions for $\delta\vec{M}$ at the film surfaces, might play a role in magnetization reversal; we will not consider this possibility in the present work. (Note also that in our model for the microstructure of a polycrystalline film, local anisotropy forces are constant across the film thickness, precluding z-dependent dispersion-induced fluctuations of $\vec{M}$.)

B. EQUATIONS OF MOTION

The gyromagnetic equation for the magnetization $\vec{M}(\vec{r}, t)$ of a ferromagnet, as given by (I-2) without damping, is

$$\frac{\partial \vec{M}(\vec{r}, t)}{\partial t} = -\gamma \vec{T}_{eff}(\vec{r}, t) \tag{V-1}$$

where

$$\vec{T}_{eff} = \vec{M}(\vec{r}, t) \times \vec{H}_{eff}(\vec{r}, t) \quad . \tag{V-2}$$

Separating the spatially varying part of $\vec{M}$ by (I-1b) and $\vec{H}_{eff}$ by

$$\vec{H}_{eff} = \vec{h}_o(t) + \delta\vec{H}(\vec{r}, t) \tag{V-3}$$

[where $\vec{h}_o(t)$ is the spatial average of $\vec{H}_{eff}(\vec{r}, t)$], and taking a spatial average of (V-1), we find

$$\frac{d\vec{m}_o}{dt} = -\gamma \vec{m}_o \times \vec{h}_o - \gamma \langle \delta\vec{M} \times \delta\vec{H} \rangle \quad . \tag{V-4a}$$

This equation for the uniform mode, with the second term on the right side replaced by a phenomenological damping term, was discussed in Sec. IV. The nonlinear reaction of spin waves on the uniform mode is described by this second term and will be computed in Sec. VI. Subtracting (V-4a) from (V-1), we obtain the dynamic equation for fluctuations of the magnetization

$$\frac{\partial}{\partial t} \delta\vec{M} = -\gamma(\vec{m}_o \times \delta\vec{H} - \vec{h}_o \times \delta\vec{M}) - \gamma(\delta\vec{M} \times \delta\vec{H} - \langle \delta\vec{M} \times \delta\vec{H} \rangle) \quad . \tag{V-4b}$$

Since (as we shall find) $\delta\vec{H}$ is proportional to $\delta\vec{M}$ (or, strictly speaking, to an integral operator on $\delta\vec{M}$), the assumption (I-1a) enables us to neglect, to a first approximation, the second term on the right side of (V-4b), which contains the effects of interactions among the components of $\delta\vec{M}$.

This becomes clearer when we expand $\delta\vec{M}$ and $\delta\vec{H}$ in Fourier series:

$$\delta\vec{M}(\vec{r}, t) = \sum_{\vec{k} \neq 0} \vec{m}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{r}} \tag{V-5a}$$

$$\delta\vec{H}(\vec{r}, t) = \sum_{\vec{k} \neq 0} \vec{h}_{\vec{k}}(t)\, e^{i\vec{k}\cdot\vec{r}} \quad . \tag{V-5b}$$

Then (V-4a) becomes

$$\frac{d\vec{m}_o}{dt} = -\gamma \vec{m}_o \times \vec{h}_o - \gamma \sum_{\vec{k} \neq 0} \vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}} \tag{V-6a}$$

and the $\vec{k} \neq 0$ components of (V-4b) are

$$\frac{d\vec{m}_{\vec{k}}}{dt} = -\gamma(\vec{m}_o \times \vec{h}_{\vec{k}} - \vec{h}_o \times \vec{m}_{\vec{k}}) - \gamma \sum_{\vec{k}' \neq \vec{k}, 0} \vec{m}_{\vec{k}-\vec{k}'} \times \vec{h}_{\vec{k}'} \quad . \tag{V-6b}$$

We will show below that the $\vec{k}^{th}$ component of the field is related to the $\vec{k}^{th}$ component of the magnetization by a field dyadic $\overleftrightarrow{\mathcal{H}}_{\vec{k}}$:†

$$\vec{h}_{\vec{k}}(t) = \overleftrightarrow{\mathcal{H}}_{\vec{k}}(t) \cdot \frac{\vec{m}_{\vec{k}}(t)}{M_o} \quad . \tag{V-7}$$

Then the same iterative procedure that was used to solve (III-23a, b) may be applied to (V-6a, b), with the resultant first approximations

$$\frac{d\vec{m}_o}{dt} = -\gamma \vec{m}_o \times \vec{h}_o \tag{V-8a}$$

and

$$\frac{d\vec{m}_{\vec{k}}}{dt} = -\gamma(\vec{m}_o \times \vec{h}_{\vec{k}} - \vec{h}_o \times \vec{m}_{\vec{k}}) \quad . \tag{V-8b}$$

We use a coordinate system based on $\vec{m}_o$, as shown in Fig. 7, in which the spatial components of $\vec{m}_{\vec{k}}$ are written

$$\vec{m}_{\vec{k}} = M_o(\vec{i}_m m_{\vec{k}} + \vec{i}_\Theta \Theta_{\vec{k}} + \vec{i}_\varphi \varphi_{\vec{k}}) \equiv M_o(m_{\vec{k}}, \Theta_{\vec{k}}, \varphi_{\vec{k}}) \quad . \tag{V-9}$$

With the spatial components of $\vec{h}_{\vec{k}}$ given by

$$\vec{h}_{\vec{k}} = (h^m_{\vec{k}}, h^\Theta_{\vec{k}}, h^\varphi_{\vec{k}}) \qquad (\text{all } \vec{k}) \tag{V-10}$$

and using the relations

$$\dot{\vec{i}}_m = (0, \dot{\Theta}_o, \dot{\varphi}_o \sin\Theta_o) \tag{V-11a}$$

$$\dot{\vec{i}}_\Theta = (-\dot{\Theta}_o, 0, \dot{\varphi}_o \cos\Theta_o) \tag{V-11b}$$

$$\dot{\vec{i}}_\varphi = (-\dot{\varphi}_o \sin\Theta_o, -\dot{\varphi}_o \cos\Theta_o, 0) \tag{V-11c}$$

(which are most easily found by inspection of Fig. 7), we obtain for the $\vec{i}_m$-component of (V-8b)

$$\frac{dm_{\vec{k}}}{dt} = \frac{d\Theta_o}{dt}\Theta_{\vec{k}} + \frac{d\varphi_o}{dt}\sin\Theta_o \varphi_{\vec{k}} - \gamma(h^\varphi_o \Theta_{\vec{k}} - h^\Theta_o \varphi_{\vec{k}}) \quad . \tag{V-12}$$

---

† If spatially varying forces such as local anisotropy are present, (V-7) has an additional term $M_a^{-1} \sum_{\vec{k}'} \overleftrightarrow{\mathcal{H}}_{\vec{k}-\vec{k}'} \cdot \vec{m}_{\vec{k}'}$, and the components are coupled.

However, (V-8a), in terms of the field components (V-10), becomes

$$\frac{d\Theta_o}{dt} = \gamma h_o^{\varphi} \tag{V-13a}$$

$$\frac{d\varphi_o}{dt} \sin\Theta_o = -\gamma h_o^{\Theta} \tag{V-13b}$$

and inserting these in (V-12) we find

$$\frac{dm_{\vec{k}}}{dt} = 0 \quad . \tag{V-14}$$

Since $m_{\vec{k}}$ is initially zero,

$$m_{\vec{k}}(t) = 0 \tag{V-15}$$

in our first (linear) approximation. [We could also have obtained (V-15) from the constraint $|\vec{M}| = \text{const.} = M_o$, which follows from (V-1) and (V-2).] From (V-13a, b) and (V-15), the $\vec{i}_\Theta$ and $\vec{i}_\varphi$ components of (V-8b) are then

$$\frac{d\Theta_{\vec{k}}}{dt} = -\gamma(h_o^m + h_o^{\Theta} \cot\Theta_o)\, \varphi_{\vec{k}} + \gamma h_{\vec{k}}^{\varphi} \tag{V-16a}$$

$$\frac{d\varphi_{\vec{k}}}{dt} = \gamma(h_o^m + h_o^{\Theta} \cot\Theta_o)\, \Theta_{\vec{k}} - \gamma h_{\vec{k}}^{\Theta} \quad . \tag{V-16b}$$

The exchange contribution to $\vec{h}_{\vec{k}}$ is found from (I-4)

$$\vec{H}_e = \frac{2A}{M_o^2} \nabla^2 \vec{M} \tag{V-17}$$

with Fourier components

$$\left(\vec{h}_{\vec{k}}\right)_e = -\frac{2A}{M_o^2} k^2 \vec{m}_{\vec{k}} \quad . \tag{V-18}$$

The magnetostatic contribution is given in the TFA by (II-24b):

$$\left(\vec{h}_{\vec{k}}\right)_m = -4\pi \left[\frac{\vec{k}\vec{k}}{k^2} \tilde{\chi}(kL) + \vec{i}_z \vec{i}_z \chi(kL)\right] \cdot \vec{m}_{\vec{k}} \tag{V-19a}$$

where

$$\chi(\kappa) = \kappa^{-1} e^{-\kappa} \sinh\kappa \qquad (= 1 - \kappa + \ldots \text{ for } \kappa << 1) \tag{V-19b}$$

$$\tilde{\chi}(\kappa) = 1 - \chi(\kappa) \qquad (= \kappa - \tfrac{2}{3} \kappa^2 + \ldots \text{ for } \kappa << 1) \quad . \tag{V-19c}$$

Finally, the effective uniaxial anisotropy field [see (III-2)] is

$$\vec{H}_a = H_K \vec{i}_x \vec{i}_x \cdot \frac{\vec{M}}{M_o} \tag{V-20}$$

with Fourier components

$$(\vec{h}_o)_a = H_K \vec{i}_x \vec{i}_x \cdot \vec{i}_m \tag{V-21a}$$

$$\left(\vec{h}_{\vec{k}}\right)_a = H_K \vec{i}_x \vec{i}_x \cdot \frac{\vec{m}_{\vec{k}}}{M_o} \quad . \tag{V-21b}$$

We see from (V-18), (V-19a), and (V-21b) that $\vec{h}_{\vec{k}}$ can indeed be expressed in the form (V-7), with

$$\overleftrightarrow{\mathcal{H}}_{\vec{k}} = -4\pi M_o\left(\tilde{\chi}\,\frac{\vec{k}\vec{k}}{k^2} + \chi\,\vec{i}_z\vec{i}_z\right) - \frac{2A}{M_o}\,k^2\overleftrightarrow{I} + H_K \vec{i}_x \vec{i}_x \tag{V-22}$$

where $\overleftrightarrow{I}$ is the idemfactor, or unit dyadic. The vectors in (V-22), in the (m, Θ, φ) coordinate system, are

$$\frac{\vec{k}}{k} \equiv \vec{\xi}(\Phi) = [\sin\Theta_o \cos(\varphi_o - \Phi), \cos\Theta_o \cos(\varphi_o - \Phi), -\sin(\varphi_o - \Phi)] \tag{V-23a}$$

$$\vec{i}_z = (\cos\Theta_o, -\sin\Theta_o, 0) \tag{V-23b}$$

$$\vec{i}_x = \vec{\xi}(0) \tag{V-23c}$$

where the wave vector $\vec{k}$ lies in the film plane at an angle Φ to the x-axis. We also note that (V-16a,b) contain the factor

$$h_o^m + h_o^\Theta \cot\Theta_o = \csc\Theta_o h_o^p \tag{V-24}$$

where h_o^p is the component of the uniform field along the projection of $\vec{m}_o$ onto the film plane.

It is convenient to write (V-16a,b) in the form

$$\frac{d\Theta_{\vec{k}}}{dt} = -\omega_{\varphi\Theta}\Theta_{\vec{k}} - \omega_{\varphi\varphi}\varphi_{\vec{k}} \tag{V-25a}$$

$$\frac{d\varphi_{\vec{k}}}{dt} = \omega_{\Theta\Theta}\Theta_{\vec{k}} + \omega_{\Theta\varphi}\varphi_{\vec{k}} \quad . \tag{V-25b}$$

For the case of an external field $\vec{H}$ in the film plane at an angle β to the x-axis, and $\vec{m}_o$ nearly in the plane such that

$$|\psi_o| << 1 \qquad (\psi_o = \Theta_o - \tfrac{\pi}{2}) \tag{V-26}$$

we find, with the help of (V-21a), (V-22), (V-23a-c), and (V-24),

$$\omega_{\Theta\Theta} = \omega_h \cos(\varphi_o - \beta) + \omega_a \cos^2\varphi_o + \omega_e L^2k^2 + \omega_m [\tilde{\chi}\psi_o^2 \cos^2(\varphi_o - \Phi) + \chi] \tag{V-27a}$$

$$\omega_{\varphi\varphi} = \omega_h \cos(\varphi_o - \beta) + \omega_a \cos 2\varphi_o + \omega_e L^2k^2 + \omega_m \tilde{\chi} \sin^2(\varphi_o - \Phi) \tag{V-27b}$$

$$\omega_{\Theta\varphi} = \omega_{\varphi\Theta} = -\frac{1}{2}\,\psi_o\,[-\omega_a \sin 2\varphi_o + \omega_m \tilde{\chi} \sin 2(\varphi_o - \Phi)] \tag{V-27c}$$

where

$$\omega_h = \gamma\,|\vec{H}| \tag{V-28a}$$

$$\omega_a = \gamma H_K \tag{V-28b}$$

$$\omega_e = \gamma\,\frac{2A}{M_o L^2} \tag{V-28c}$$

$$\omega_m = \gamma 4\pi M_o \quad . \tag{V-28d}$$

Note that our definition of the exchange frequency ω_e is not the conventional one, since it is based on the half-thickness of the film L, rather than on the lattice constant ℓ. Equations (V-13a, b), which describe the uniform mode, now become

$$\frac{d\psi_o}{dt} = -\omega_h \sin(\varphi_o - \beta) - \omega_a\,\frac{1}{2}\,\sin 2\varphi_o \tag{V-29a}$$

$$\frac{d\varphi_o}{dt} = \psi_o\,[\omega_h \cos(\varphi_o - \beta) + \omega_a \cos^2\varphi_o + \omega_m] \quad . \tag{V-29b}$$

The coupled, linear equations (V-25a,b), in which the coefficients ω_{ij} are, in general, time dependent through the time dependence of φ_o and ψ_o, describe the motion of the components of $\delta\vec{M}$ out of the film plane ($\Theta_{\vec{k}}$) and in the plane ($\varphi_{\vec{k}}$). In Secs. V-C and V-D we will find quasistatic spin-wave solutions (with ψ_o, φ_o constant) and transient solutions (with $\vec{H}$ turned on at $t = 0$).

C. QUASISTATIC SPECTRUM

Here, we assume that there exists a time interval over which $\psi_o(t)$ and $\varphi_o(t)$ may be considered constants while $\Theta_{\vec{k}}(t)$ and $\varphi_{\vec{k}}(t)$ are rapidly varying. Then solutions of (V-25a, b) are spin waves with time dependence

$$e^{i\omega_{\vec{k}} t} \tag{V-30}$$

where the eigenfrequencies $\omega_{\vec{k}}$ are given by

$$\omega_{\vec{k}}^2 = \omega_{\Theta\Theta}\,\omega_{\varphi\varphi} - \omega_{\Theta\varphi}^2 \quad . \tag{V-31}$$

We first examine the spectrum (V-31) for the usual resonance situation of oscillations about a stable equilibrium. Then ψ_o and φ_o are given by their equilibrium values $\check{\psi}_o$ and $\check{\varphi}_o$ [which may be found from (V-29a,b)]

$$\check{\psi}_o = 0 \tag{V-32a}$$

$$H \sin(\check{\varphi}_o - \beta) + \frac{1}{2}\,H_K \sin 2\check{\varphi}_o = 0 \tag{V-32b}$$

with the stability condition

$$H\cos(\check{\varphi}_o - \beta) + H_K \cos 2\check{\varphi}_o > 0 \quad . \tag{V-32c}$$

Inserting $\check{\psi}_o$ and $\check{\varphi}_o$ in (V-27a-c), (V-31) becomes

$$\omega_{\vec{k}}^2 = [\omega_h \cos(\check{\varphi}_o - \beta) + \omega_a \cos^2\check{\varphi}_o + \omega_e L^2k^2 + \omega_m \chi]$$
$$\times [\omega_h \cos(\check{\varphi}_o - \beta) + \omega_a \cos 2\check{\varphi}_o + \omega_e L^2k^2 + \omega_m \tilde{\chi} \sin^2(\check{\varphi}_o - \Phi)] \quad . \tag{V-33a}$$

The eigenfrequency of the uniform mode may be found by expanding (V-29a,b) about $\psi_o = 0$, $\varphi_o = \check{\varphi}_o$, or from (V-33a,b) in the limit $k \to 0$, and is given by

$$\omega_o^2 = [\omega_h \cos(\check{\varphi}_o - \beta) + \omega_a \cos^2\check{\varphi}_o + \omega_m][\omega_h \cos(\check{\varphi}_o - \beta) + \omega_a \cos 2\check{\varphi}_o] \quad . \tag{V-33b}$$

If the external field is much greater than the anisotropy field, $\check{\varphi}_o = \beta$, and (V-33a,b) become (for $\vec{H}$ along $\vec{i}_x$)

$$\boxed{\omega_{\vec{k}}^2 = (\omega_h + \omega_e L^2k^2 + \omega_m \chi)(\omega_h + \omega_e L^2k^2 + \omega_m \tilde{\chi} \sin^2\Phi)} \tag{V-34a}$$

$$\boxed{\omega_o^2 = \omega_h(\omega_h + \omega_m)} \tag{V-34b}$$

which are shown in Fig. 8 for a TPF in a field of 800 oe. In the infinite medium limit $kL \to \infty$

$$\omega_{\vec{k}}^2 \to (\omega_h + \omega_e L^2k^2)(\omega_h + \omega_e L^2k^2 + \omega_m \sin^2\Phi) \tag{V-35}$$

which is the dispersion relation for a thin disk magnetized in its plane as given by Callen.[43] Extending this spectrum to $\vec{k} = 0$ (see Fig. 8), we see that the uniform mode lies at the top of the spin-wave manifold ($\Phi = \pi/2$), in contrast to the TFA spectrum (V-34a), which collapses to a point as $\vec{k} \to 0$. We also show in Fig. 8 the magnetostatic modes found by Damon and Eshbach.[48] Since the location of the uniform mode relative to the spin-wave manifold determines the coupling of the uniform mode to spin waves,[43,46] the distortion of the infinite medium spectrum at long wavelengths by magnetostatic fields will have profound effects on resonance phenomena in thin films. Such effects have been observed by Comly, Penney, and Jones[51] at high microwave power levels; their results show that the bottom of the manifold is indeed correctly described by Eq. (V-34a) with $\Phi = 0$.[52]

Spin-wave solutions of the linearized equations of motion (V-25a,b) and (V-29a,b) also exist about unstable equilibria of the uniform mode ($\hat{\psi}_o, \hat{\varphi}_o$), which are associated with magnetization reversal processes, and which may be found from Eqs. (V-32a-c) with the inequality of (V-32c) reversed. The dynamic trajectory of the uniform mode may not pass through the point $\psi_o = 0$, but because of the condition (V-26) [see (IV-51) and (IV-24)] this makes very little difference in the dispersion relation for $\omega_{\vec{k}}$. It has been pointed out[27,53,54] that the spin-wave eigenfrequencies will then be imaginary for a certain set of wave vectors, and that unstable solutions with time dependence

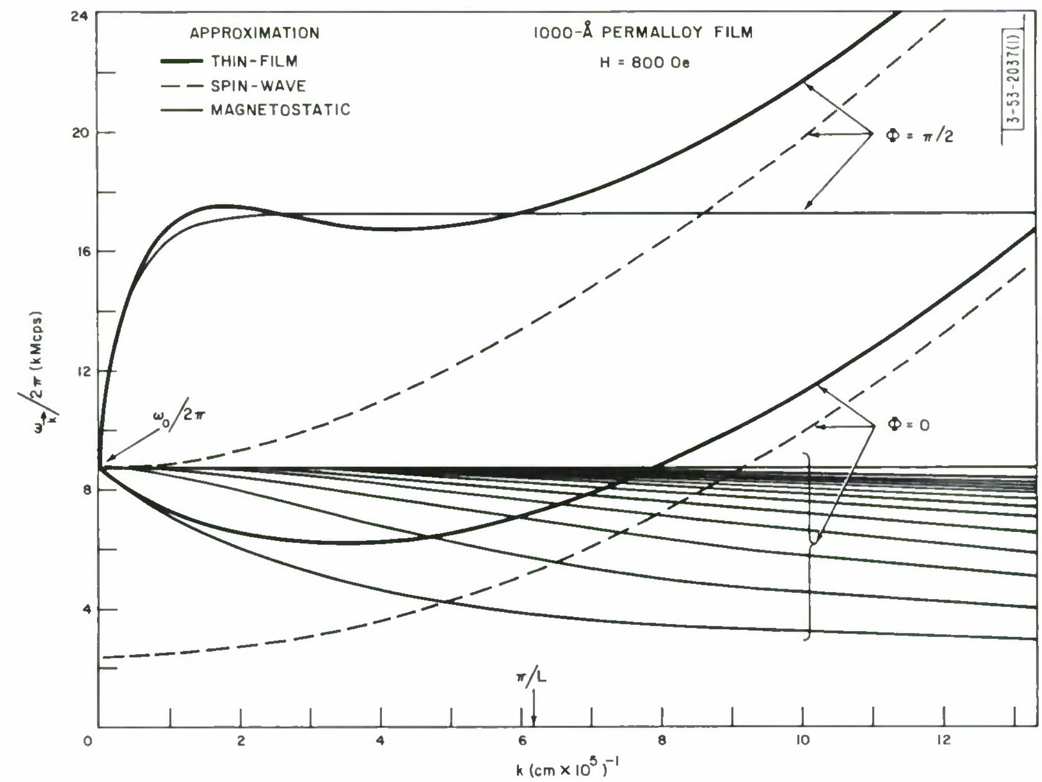

Fig. 8. Resonant modes of a thin film in a planar field of 800 oe in thin-film approximation (average over thickness), spin-wave approximation (neglect magnetostatic boundary conditions), and magnetostatic approximation (neglect exchange fields) (for a TPF).

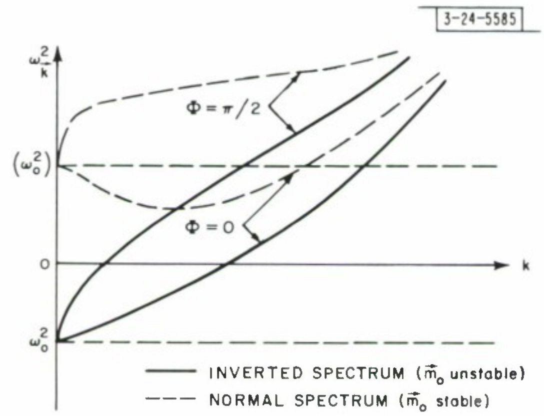

Fig. 9. Schematic illustration of inverted and normal planar mode spectra (thin-film approximation).

$$e^{\Omega_{\vec{k}} t} \qquad (\Omega_{\vec{k}} = i\omega_{\vec{k}} > 0) \tag{V-36}$$

may be important in switching. We now show that the growth time (for increase in amplitude by a factor e) of the fastest growing spin waves is approximately the same as the coherent rotation reversal time, so that switching is completed before any spin-wave amplitudes increase much above their equilibrium values.

It is easy to see from the dispersion relation (V-33a) (remembering that we are only interested in low fields, i.e., $\omega_h << \omega_m$, $\omega_a << \omega_m$) that unstable spin waves ($\omega_{\vec{k}}^2 < 0$) will exist providing

$$-(\omega_a \sin^2 \hat{\varphi}_o + \omega_m \chi) < \omega_h \cos(\hat{\varphi}_o - \beta) + \omega_a \cos 2\hat{\varphi}_o$$

$$+ \omega_e L^2 k^2 < -\omega_m \tilde{\chi} \sin^2(\hat{\varphi}_o - \Phi) \quad . \tag{V-37}$$

The fastest growing of these will have $\Phi = \hat{\varphi}_o$ (propagation along $\vec{m}_o$) and $kL << 1$, so that their growth rates are

$$\Omega_{\vec{k}} \simeq \omega_m^{1/2} [-\omega_h \cos(\hat{\varphi}_o - \beta) - \omega_a \cos 2\hat{\varphi}_o - \omega_e L^2 k^2]^{1/2} > 0 \quad . \tag{V-38}$$

In Fig. 9 we sketch the "inverted" spectrum ($\vec{m}_o$ unstable) and, for comparison, the normal spectrum. From (V-38) we have the important result that the growth rate of spin waves in a thin film in unstable equilibrium in low fields ($<< 4\pi M_o$) is bounded by the growth rate of the uniform mode, or

$$\underset{(\vec{k} \neq 0)}{\Omega_{\vec{k}}} < \Omega_o = \gamma(4\pi M_o)^{1/2} [-H_p \cos(\hat{\varphi}_o - \beta_p) - H_b \cos(\hat{\varphi}_o - \beta_b) - H_K \cos 2\hat{\varphi}_o]^{1/2} \tag{V-39}$$

where we have generalized the external field $\vec{H}$ as described on p. 34. This result, which strongly suggests that spin-wave instabilities will not greatly influence the behavior of the uniform mode, may be made more precise by comparing the maximum growth rate for spin waves, Ω_o, with the inverse switching time, t_s^{-1}, obtained in Sec. IV. From (IV-47), and using the approximation suggested on p. 42, we find

$$t_s^{-1} = \gamma\, [4\pi M_o (H_p - H_{pt})]^{1/2}\, J(\beta_p, \beta_b) \tag{V-40}$$

where $J \sim 1$, except for $H_b = 0$ and $\beta_p = \pi$,† and reduces to $1/[K(\sin\frac{1}{2}\beta)]$ for $H_b = H_K = 0$. The pulse threshold field H_{pt} is the field at the transition from stable to unstable equilibrium, or

$$H_{pt} \cos(\hat{\varphi}_o - \beta_p) + H_b \cos(\hat{\varphi}_o - \beta_b) + H_K \cos 2\hat{\varphi}_o = 0 \quad . \tag{V-41}$$

Inserting (V-41) in (V-39) we obtain

---

† $\beta = \pi$ is a pathalogical case of balanced unstable equilibrium, and we shall exclude it from our treatment.

$$\Omega_o = \gamma\,[-4\pi M_o(H_p - H_{pt})\cos(\hat{\varphi}_o - \beta)]^{1/2} \tag{V-42}$$

which is never much greater than t_s^{-1}.

The results obtained in Sec. V-C are valid whatever the origin of the magnetic fluctuations – thermal agitation, inhomogeneities, anisotropy dispersion, etc., – and are, therefore, completely independent of our random anisotropy model. In the following pages we shall consider transient spin waves in a step function field $\vec{H}_p$, taking as initial values the dispersion-induced ripple components of Sec. III.

D. TRANSIENT RESPONSE

We first rewrite (V-25a, b) with the normalized variables [see (IV-22) to (IV-24)]

$$\tau = \sqrt{\omega_h \omega_m}\, t = \gamma \sqrt{4\pi M_o H_p}\, t \tag{V-43a}$$

$$\sigma_o = \epsilon^{-1} \psi_o \tag{V-43b}$$

where

$$\epsilon = \sqrt{\frac{\omega_h}{\omega_m}} = \sqrt{\frac{H_p}{4\pi M_o}} = O(10^{-2}) \quad . \tag{V-43c}$$

These variables have been chosen such that during magnetization reversal $\dot{\varphi}_o$, $|\dot{\sigma}_o|$, and σ_o are $O(1)$, as may be inferred from (IV-38a, b) and (IV-50). (The dot indicates differentiation with respect to τ.) The equations of motion of the spin-wave components are now

$$\dot{\Theta}_{\vec{k}} = -G_{\varphi\Theta}\Theta_{\vec{k}} - G_{\varphi\varphi}\varphi_{\vec{k}} \tag{V-44a}$$

$$\dot{\varphi}_{\vec{k}} = G_{\Theta\Theta}\Theta_{\vec{k}} + G_{\Theta\varphi}\varphi_{\vec{k}} \tag{V-44b}$$

with initial conditions

$$\Theta_{\vec{k}}(0) = 0 \tag{V-45a}$$

$$\varphi_{\vec{k}}(0) \quad \text{given by (III-35)} \tag{V-45b}$$

where

$$G_{\Theta\Theta} = \epsilon^{-1}\chi + \epsilon\,\{\sigma_o^2\,[\tilde{\chi}\cos^2(\varphi_o - \Phi) - \chi] + (\Lambda\lambda_e^2 k^2 + \sin^2\varphi_o)\,h_p^{-1} - \rho'(\varphi_o)\} + O(\epsilon^2) \tag{V-46a}$$

$$G_{\varphi\varphi} = \epsilon^{-1}\tilde{\chi}\sin^2(\varphi_o - \Phi) + \epsilon\,[\Lambda\lambda_e^2 k^2 h_p^{-1} - \rho'(\varphi_o)] + O(\epsilon^2) \tag{V-46b}$$

$$G_{\Theta\varphi} = G_{\varphi\Theta} = -\frac{1}{2}\,\sigma_o\tilde{\chi}\sin 2(\varphi_o - \Phi) + O(\epsilon^2) \quad . \tag{V-46c}$$

Here we have written the exchange field in terms of the exchange length λ_e, defined by (III-36). The remaining parameters in (V-46a, b) are

$$\Lambda(\beta_b, h_b) = \cos 2\varphi_o(0) + h_b \cos[\beta_b - \varphi_o(0)] \tag{V-47a}$$

$$h_i \equiv \frac{H_i}{H_K} \quad (i = p, b, \text{ etc.}) \tag{V-47b}$$

$$-\rho'(\varphi_o) = \cos(\varphi_o - \beta_p) + \frac{h_b}{h_p}\cos(\varphi_o - \beta_b) + \frac{1}{h_p}\cos 2\varphi_o \tag{V-47c}$$

[see (III-32) and (IV-21c)]. The uniform mode is described by (IV-27a,b) without damping, or

$$\dot{\varphi}_o = \sigma_o \tag{V-48a}$$

$$\dot{\sigma}_o = \rho(\varphi_o) = -\sin(\varphi_o - \beta_p) - \frac{h_b}{h_p}\sin(\varphi_o - \beta_b) - \frac{1}{2h_p}\sin 2\varphi_o \tag{V-48b}$$

with initial conditions

$$h_b \sin[\varphi_o(0) - \beta_b] + \frac{1}{2}\sin 2\varphi_o(0) = 0 \tag{V-48c}$$

$$\sigma_o(0) = 0 \tag{V-48d}$$

so that

$$\rho[\varphi_o(0)] = \sin[\beta_p - \varphi_o(0)] \equiv \rho_o > 0 \quad . \tag{V-48e}$$

For the case (IV-19a, c) of a hard axis bias field and an easy axis pulse field,

$$\rho(\varphi_o) = \sin\varphi_o + \frac{h_\perp}{h_{||}}\cos\varphi_o - \frac{1}{2h_{||}}\sin 2\varphi_o \tag{V-49a}$$

$$\rho_o = h_\perp \quad . \tag{V-49b}$$

The finite rise time of the pulse field will not be considered in this treatment; thus, $\rho(\tau) = 0$ for $\tau < 0$.

We next eliminate $\Theta_{\vec{k}}$ from (V-44a,b), obtaining

$$\ddot{\varphi}_{\vec{k}} - \frac{\dot{G}_{\Theta\Theta}}{G_{\Theta\Theta}}\dot{\varphi}_{\vec{k}} + \left[G_{\Theta\Theta}G_{\varphi\varphi} - G^2_{\Theta\varphi} + G_{\Theta\varphi}\frac{\dot{G}_{\Theta\Theta}}{G_{\Theta\Theta}} - \dot{G}_{\Theta\varphi}\right]\varphi_{\vec{k}} = 0 \tag{V-50a}$$

with initial conditions

$$\dot{\varphi}_{\vec{k}}(0) = 0 \tag{V-50b}$$

$$\varphi_{\vec{k}}(0) \neq 0 \quad . \tag{V-50c}$$

As can be seen from (III-35), and as discussed in Sec. III-D, short wavelength ripple components are suppressed by exchange forces. We therefore confine ourselves to the long wavelength end of the spectrum

$$k \lesssim \lambda_e^{-1} \tag{V-51}$$

and in Sec. VI we will show that the main contribution to the spin-wave torque on the uniform mode comes from components with $k \sim \lambda_e^{-1}$. With this restriction, the coefficients of powers of ϵ in the G_{ij}'s of (V-46a-c) (and their time derivatives) are all $O(1)$.

We are now in a position to greatly simplify Eq. (V-50a). Inserting the coefficients G_{ij} from (V-46a-c), and dropping negligible terms, we find

$$\ddot{\varphi}_{\vec{k}} + \epsilon^2 \chi^{-1} a_1(\tau)\, \dot{\varphi}_{\vec{k}} + \epsilon^{-2} \chi \tilde{\chi} \sin^2 [\varphi_o(\tau) - \Phi]\, \varphi_{\vec{k}} = 0 \tag{V-52}$$

where $a_1(\tau) = O(1)$. We next define a large parameter λ by

$$\lambda = \epsilon^{-1} \sqrt{\chi \tilde{\chi}} \quad . \tag{V-53}$$

For $kL << 1$

$$\lambda = \epsilon^{-1} (kL)^{1/2} \left(1 - \frac{5}{6}\, kL + \ldots\right) \tag{V-53-1}$$

and for a TPF with $k = \lambda_e^{-1}$ and $H_p = 5$ oe, $\lambda \simeq 20$. Equation (V-52) may now be written

$$\ddot{\varphi}_{\vec{k}} + \lambda^{-2}\, a_2(\tau)\, \dot{\varphi}_{\vec{k}} + [\lambda f(\tau)]^2\, \varphi_{\vec{k}} = 0 \tag{V-54}$$

where

$$a_2(\tau) = \tilde{\chi} a_1(\tau) \qquad (|a_2| \leqslant |a_1|) \tag{V-55a}$$

$$f(\tau) = \sin [\varphi_o(\tau) - \Phi] \quad . \tag{V-55b}$$

Clearly the "damping" term $\lambda^{-2} a_2 \dot{\varphi}_{\vec{k}}$ is negligible compared to the "potential" term $\lambda^2 f^2 \varphi_{\vec{k}}$, since $|\dot{\varphi}_{\vec{k}}| \sim \lambda |f \varphi_{\vec{k}}|$. With the notation

$$\varphi(\tau) = \frac{\varphi_{\vec{k}}(\tau)}{\varphi_{\vec{k}}(0)} \tag{V-56}$$

our problem is now reduced to solving the equation

$$\ddot{\varphi} + [\lambda f(\tau)]^2\, \varphi = 0 \tag{V-57}$$

with initial conditions

$$\varphi(0) = 1 \tag{V-58a}$$

$$\dot{\varphi}(0) = 0 \tag{V-58b}$$

where $f(\tau)$ is given by (V-55b), φ_o is determined by (V-48a-e), and $\lambda >> 1$.

The potential $(\lambda f)^2$ has arisen essentially from volume poles which, in equilibrium, strongly attenuate ripple components propagating in all directions except along $\pm \vec{m}_o$; we recall from Sec. III-D that the zone of nonattenuated components is extremely narrow. (See Fig. 5.) In fast magnetization reversal, however, volume magnetostatic fields quickly build up to large values, since components initially propagating along $\pm \vec{m}_o$ "have the rug pulled out from under them" when $\vec{m}_o$ rotates, and go through a transient state resembling the energetically unfavorable transverse ripple. This is shown schematically in Fig. 10. On the other hand, although the initial very small amplitude components with $\vec{k}$ in all directions other than $\pm \vec{m}_o$ do go through

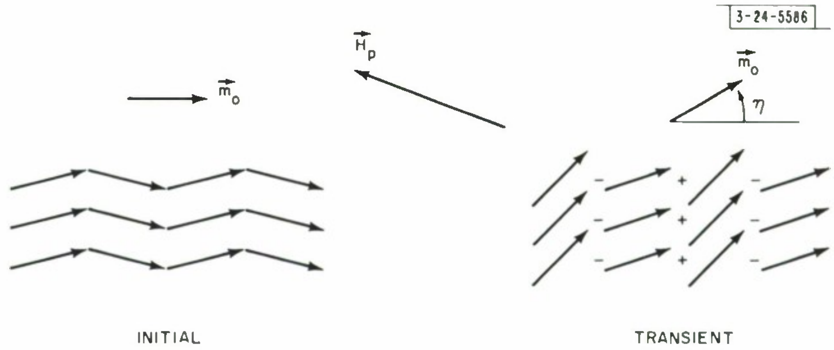

Fig. 10. Schemotic illustrotion of initiol ond transient spin-wove stotes (LMR) in rototionol switching, showing creotion of volume poles.

a transient state of low magnetostatic energy, their amplitudes will not grow much above equilibrium values unless very fast relaxation can occur (relaxation time $\ll t_s$). However, there is no evidence for such a rapid process, and in any event we are neglecting all relaxation in this treatment. Thus, we can conclude that the primary effects in magnetization reversal will come from spin waves with $\Phi \simeq \varphi_o(0)$ and $\Phi \simeq \varphi_o(0) + \pi$, that is, the initial LMR. But since $\delta\vec{M}$ is real, we see from (V-5a) that

$$\vec{m}_{-\vec{k}} = \vec{m}^*_{\vec{k}} \tag{V-59}$$

and it is therefore sufficient to solve (V-57) for $\Phi \simeq \varphi_o(0)$.

The presence of the large parameter λ in (V-57), coupled with the fact that $f(\tau)$ is not rapidly varying, suggests the WKB method, for which we refer to Appendix C. Our main interest is in the asymptotic solution (C-14), which is valid except near free points, at which $f(\tau) = 0$. The magnitude and phase of this solution is determined from the initial conditions (V-58a,b); but for the components of interest, $\tau = 0$ is at or very near a free point. Therefore, a solution joining the asymptotic region to the neighborhood of the free point must be found. We first make the substitutions

$$\eta(\tau) = \varphi_o(\tau) - \varphi_o(0) \tag{V-60a}$$

$$\Psi = \Phi - \varphi_o(0) \tag{V-60b}$$

so that (V-55b) becomes

$$f(\tau) = \sin[\eta(\tau) - \Psi] \tag{V-61}$$

with $\eta(\tau)$ determined from (V-48a-e), which become

$$\dot{\eta}(\tau) = \sigma_o(\tau) \tag{V-62a}$$

$$\dot{\sigma}_o(\tau) = \rho(\eta) \tag{V-62b}$$

$$\eta(0) = 0 \tag{V-62c}$$

$$\sigma_o(0) = 0 \tag{V-62d}$$

$$\rho(0) = \rho_o > 0 \quad . \tag{V-62e}$$

The asymptotic generating function for $[\lambda f(\tau)]^2$ is given by (C-15)

$$g(\tau) = \lambda \int_{\tau_o}^{\tau} f(\xi)\, d\xi \tag{V-63}$$

where τ_o is a constant, as yet unspecified. We examine g in the vicinity of the free point $\eta = \Psi$, with

$$|\Psi| << 1 \tag{V-64}$$

in accord with the previous discussion.

Consider first the case $\Psi \geqslant 0$, for which we let

$$\eta(\tau_o) = \Psi \quad . \tag{V-65}$$

Since η is initially zero and Ψ is small, the free point τ_o may be found by expanding $\eta(\tau)$ in a Taylor's series about $\tau = 0$, with the result

$$\tau_o = \sqrt{\frac{2\Psi}{\rho_o}} \tag{V-66a}$$

provided

$$\Psi << \rho_o \quad . \tag{V-66b}$$

Now we let

$$x = \tau - \tau_o \tag{V-67}$$

and expand f about τ_o, obtaining

$$f(x) = \rho_o\left(\tau_o x + \frac{x^2}{2!} + \rho_1\tau_o \frac{x^3}{3!} + \rho_1 \frac{x^4}{4!} + \ldots\right) \tag{V-68a}$$

where

$$\rho_1 = \rho'(0) \quad . \tag{V-68b}$$

Inserting this result into (V-63), we find

$$g(x) = \lambda\rho_o\left(\tau_o \frac{x^2}{2!} + \frac{x^3}{3!} + \rho_1\tau_o \frac{x^4}{4!} + \rho_1 \frac{x^5}{5!} + \ldots\right) \quad . \tag{V-69}$$

For the limiting case $\Psi = 0$ ($\tau_o = 0$), the x^3 term of (V-69) will approximate g for x in the range $\Re_2$, given by

$$x^2 << \frac{20}{|\rho_1|} \quad . \tag{V-70}$$

Then, following the argument of pp. 79-80, we find that the approximate solution to (V-57) in $\Re_2$ is

$$\varphi(x) \simeq \left[\frac{g(x)}{\lambda f(x)}\right]^{1/2} J_{\pm 1/6}[g(x)] \quad . \tag{V-71}$$

The asymptotic expansion of φ [given by (C-11) and valid in $\Re_\infty$ defined by (C-10b)] is valid in part of $\Re_2$ if the inequality (C-23) is satisfied for some x_2 in $\Re_2$; with $\nu = 1/6$ and $f_m = \rho_o/2$, (C-23) becomes

$$|x_2|^3 >> \frac{2}{3\lambda\rho_o} \quad . \tag{V-72}$$

Then, combining (V-70) and (V-72) we arrive at the condition for overlap of $\mathcal{R}_2$ and $\mathcal{R}_\infty$

$$\lambda >> \frac{1}{12\rho_o}\left(\frac{|\rho_1|}{5}\right)^{3/2} \tag{V-73}$$

which is well satisfied except as $\rho_o \to 0$, the excluded case of a pulse field antiparallel to $\vec{m}_o(0)$. (See footnote on p. 51.)

Now, suppose $\Psi > 0$. Under what conditions will the solution (V-71) still join the asymptotic region $\mathcal{R}_\infty$ to the vicinity of the initial point, $x = -\tau_o$? First, (V-73) must be satisfied; but, second, and more important, the largest value of $|x|$ for which the x^2 term of (V-69) is not negligible compared to the x^3 term must be less than the smallest value of $|x|$ in $\mathcal{R}_\infty$. For, otherwise, the asymptotic solution will join not with (V-71) but with a solution involving also (or exclusively) the first term of (V-68a). This condition is met if

$$\tau_o << (\lambda\rho_o)^{-1/3} \tag{V-74}$$

as may be seen from (V-69) and (V-72); using (V-66a), (V-74) becomes

$$\Psi << \frac{1}{2}\rho_o^{1/3}\lambda^{-2/3} \equiv \Psi_o << 1 \quad . \tag{V-74-1}$$

If $\Psi < 0$, τ_o as given by (V-66a) is imaginary; there is a nearly free point at $\tau = 0$, but no free point near $\eta(\tau) = 0$ for real τ. However, we may consider x, defined by (V-67), as a complex variable and by analytic continuation extend the definitions of f(x) and g(x) into the complex domain. Then all the results for $\Psi \geqslant 0$ are equally valid for $\Psi < 0$, provided x, τ_o, and Ψ are replaced by their absolute values in (V-66b), (V-70), (V-74), and (V-74-1).

Collecting our results, we have found that a solution to (V-57), valid from the neighborhood of $\tau = 0$ up to, but not including, the next free point (near $\eta = \pi$), is

$$\varphi(x) \simeq \left[\frac{g(x)}{\lambda f(x)}\right]^{1/2} \{B_+ J_{1/6}[g(x)] + B_- J_{-1/6}[g(x)]\} \tag{V-75}$$

provided the angle Ψ between the initial direction of $\vec{m}_o$ and the wave vector $\vec{k}$ of the component $\varphi_{\vec{k}}$ satisfies

$$|\Psi| << \Psi_o << 1 \quad . \tag{V-76}$$

Note that what we have found is essentially a $\Psi = 0$ solution, as it is based on the $\Psi = 0$ generating function [(V-69) with $\tau_o = 0$]. For $|\Psi| >> \Psi_o$, it is not hard to show that a solution of the form (V-75) exists with $\nu = \frac{1}{4}$. But for $|\Psi| \sim \Psi_o$, no such solution is possible because the asymptotic form must be connected to a solution in a range of x for which the x^2 and x^3 terms of (V-69) are the same order of magnitude. However, we shall show in Sec. VI that the strong inequality (V-76) does not exclude any physically important components of the ripple.

The constants $B_\pm$ are evaluated from the initial conditions (V-58a,b). At $\tau = 0$, $x = -\tau_o$, so that from (V-69) and (V-74)

$$|g|_{\tau=0} \simeq \frac{\lambda\rho_o}{3}|\tau_o|^3 << 1 \quad . \tag{V-77}$$

Then, using the expansion of $J_\nu(g)$ about $g = 0$,[55]

$$J_\nu(g) = \left(\frac{g}{2}\right)^\nu \sum_{m=0}^{\infty} \left(\frac{g}{2}\right)^{2m} \frac{(-1)^m}{m!\,\Gamma(\nu + m + 1)} \tag{V-78}$$

and retaining only the leading term of g

$$g(x) \simeq \frac{\lambda\rho_o}{6} x^3 \tag{V-79}$$

we find

$$\varphi(x) = \frac{1}{\sqrt{3}} \left[\frac{B_-}{\Gamma(\frac{5}{6})} \left(\frac{\lambda\rho_o}{12}\right)^{-1/6} + \frac{B_+}{\Gamma(\frac{7}{6})} \left(\frac{\lambda\rho_o}{12}\right)^{1/6} x + \ldots \right] \quad . \tag{V-80}$$

Then, since at $\tau = 0$

$$|x| << (\lambda\rho_o)^{-1/3} \tag{V-81}$$

the initial conditions (V-58a,b) lead to the result

$$B_+ \simeq 0 \tag{V-82a}$$

$$B_- \simeq (\lambda\rho_o)^{1/6}\, \Gamma(\tfrac{5}{6})\, (\tfrac{3}{2})^{1/3} \quad . \tag{V-82b}$$

Finally, we obtain the asymptotic expansion of $\varphi(\tau)$. The asymptotic region $\mathfrak{R}_\infty$ is given by (C-10b)

$$|g| >> \frac{1}{9} \tag{V-83a}$$

or [see (V-79)]

$$(\tfrac{3}{2}\, \lambda\rho_o)^{-1/3} << \tau < \tau_1 \tag{V-83b}$$

where

$$\eta(\tau_1) \simeq \pi + \Psi \quad . \tag{V-83c}$$

The upper bound $\tau_1 = O(\tau_s)$, and since we will not be investigating the behavior of spin waves at the end of the reversal process, there is no need to specify τ_1 precisely. From (C-11), noting that $\phi = 0$ in $\mathfrak{R}_\infty$, we have the result†

$$\varphi(\tau) \xrightarrow[\tau \text{ in } \mathfrak{R}_\infty]{} B_- \sqrt{\frac{2}{\pi\lambda f(\tau)}} \cos\left[g(\tau) - \frac{\pi}{6}\right] \tag{V-84}$$

† Except for changes in the constant B_- and the phase $\pi/6$, (V-84) is the asymptotic value of $\varphi(\tau)$ for all Ψ. A computer solution of (V-57), however, has shown that the true constant is approximately equal to B_- for $\Psi \lesssim \Psi_o$, with maximum error, at $\Psi = \Psi_o$, of 15 percent.

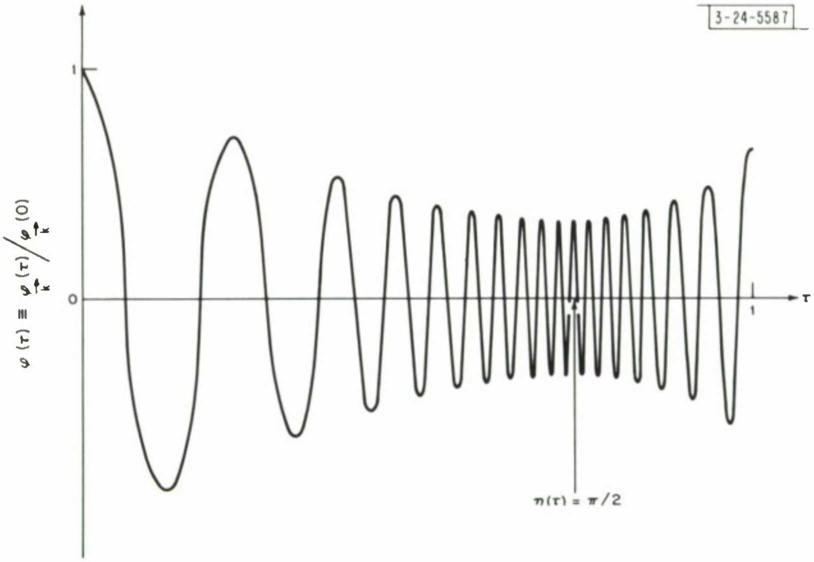

Fig. 11. Transient response of a planar spin-wave component in rotational switching.

or [using (V-56)]

$$\boxed{\varphi_{\vec{k}}(\tau) \xrightarrow[\tau \text{ in } \Re_\infty]{} \varphi_{\vec{k}}(0)\, C_\infty \lambda^{-1/3} \rho_o^{1/6} [f(\tau)]^{-1/2} \cos[g(\tau) - \tfrac{\pi}{6}]} \tag{V-85}$$

where

$$C_\infty = (\tfrac{3}{2})^{1/3} \sqrt{\tfrac{2}{\pi}}\, \Gamma(\tfrac{5}{6}) \quad . \tag{V-86}$$

Figure 11 is a sketch of $\varphi(\tau)$. The spin-wave component normal to the film plane may be found from (V-44b) and (V-56):

$$\Theta_{\vec{k}} = G_{\Theta\Theta}^{-1} [\dot{\varphi}(\tau) - G_{\Theta\varphi} \varphi(\tau)]\, \varphi_{\vec{k}}(0) \tag{V-87}$$

and with the help of (V-46a, c) and (V-53) we find

$$\boxed{\Theta_{\vec{k}}(\tau) \xrightarrow[\tau \text{ in } \Re_\infty]{} -\varphi_{\vec{k}}(0)\, C_\infty \lambda^{-1/3} \rho_o^{1/6} \sqrt{\tfrac{\tilde{\chi}}{\chi}} \{[f(\tau)]^{1/2} \sin[g(\tau) - \tfrac{\pi}{6}] + O(\lambda^{-1})\}} \quad . \tag{V-88}$$

Equations (V-85) and (V-88) comprise the main result of Sec. V-D. They show that except at the beginning and end of the reversal process, $\Psi = 0$ spin waves (LMR) precess in an elliptical path about $\vec{m}_o(\tau)$ at a frequency

$$\begin{aligned} \omega_{\vec{k}}(\tau) &= \sqrt{\omega_m \omega_h}\, \frac{g(\tau)}{\tau} \\ &= \omega_m \sqrt{\chi\tilde{\chi}}\, \frac{1}{\tau} \int_o^\tau \sin\eta(\xi)\, d\xi \quad . \end{aligned} \tag{V-89}$$

The ellipticity is given by

$$r_{\vec{k}}(\tau) = \left| \frac{\Theta_{\vec{k}}}{\varphi_{\vec{k}}} \cot\left(g - \frac{\pi}{6}\right) \right| = \sqrt{\frac{\tilde{\chi}}{\chi}} \sin \eta(\tau) \quad . \qquad \text{(V-90)}$$

[In a TPF the important set of components with $k \sim \lambda_e^{-1}$ have, at $\eta(\tau) = \pi/2$, $\omega_{\vec{k}} \sim 35 \times 10^9 \ \mathrm{sec}^{-1}$ and $r_{\vec{k}} \sim \frac{1}{2}$.] We note that the results we have obtained depend only implicitly on the uniform reversal mode, except at the very start of switching. Thus even though we have neglected damping, any real loss mechanism will require a finite time to become effective, and therefore may be included through its effect on $\varphi_o(\tau)$. In particular, the perturbation of the uniform mode by spin waves, which will be treated in the next section, need not be small for the theory to be valid; $\vec{m}_o(\tau)$ may be determined self-consistently from (V-6a), (V-85), and (V-88).

VI. EFFECT OF SPIN WAVES ON MAGNETIZATION REVERSAL

A. INTRODUCTION

Although in static equilibrium ripple has very little influence on the mean magnetization $\vec{m}_o$, the dynamic situation is quite different. As noted in Sec. V, volume magnetostatic fields attain large values during fast rotational switching because $\vec{k}$ (for the largest amplitude components) and $\vec{m}_o$ no longer remain parallel. From an energy viewpoint, this transient transverse-component ripple is created at the expense of the energy $-\vec{H}_p \cdot \vec{m}_o(0)$ which the uniform mode has acquired from the pulse field. The transient spin-wave state therefore provides a sought-after dissipation mechanism for $\vec{m}_o$, at least during the first quadrant of switching $[0 \leqslant \eta(\tau) \leqslant \pi/2$, where $\eta(\tau) = \varphi_o(\tau) - \varphi_o(0)]$. When $\eta > \pi/2$, if these components cannot lose their energy to the lattice or to other components they will return it to the uniform mode, thereby conserving energy (since magnetostatic fields are decreasing). Thus we expect damping in the first quadrant and anti-damping in the second; this is indeed what we shall find in the calculation to follow.

However, a much more striking effect is anticipated: spin-wave locking of the uniform reversal mode, which will occur if the magnetostatic energy of the intermediate state is greater than the energy available for rotational switching. Since this latter energy decreases with decreasing H_p, we may expect to find a critical field H_{pc} below which the uniform mode is locked. In Sec. VI-C, we calculate this critical field, which provides an important point of contact with experiment. We also suggest how rotational switching might take place with $H_p < H_{pc}$.

These phenomena may also be understood from a torque viewpoint. The nonlinear spin-wave reaction torque on $\vec{m}_o$, which we shall find to be predominantly z-directed, retards reversal in the first quadrant and accelerates it in the second. If at any point in the reversal process with $\vec{m}_o$ in the first quadrant the reaction torque becomes equal in absolute magnitude to the uniform reversing torque $\vec{i}_z \cdot \vec{m}_o \times \vec{h}_o$, the uniform reversal mode will become locked. We now put these ideas on a quantitative basis by computing the reaction torque, using the results of Secs. III and V.

B. NONLINEAR REACTION TORQUE

The spatially varying part of the magnetization, $\delta\vec{M}(\vec{r}, t)$, will react on the mean magnetization through the spatial average of the ensemble average of the nonlinear contribution to the effective torque. (By nonlinear, we mean second order in the small quantities $\delta\vec{M}/M_o$ and $\delta\vec{H}/M_o$; there is, of course, no linear reaction torque.) This appears in the equation of motion for $\vec{m}_o(t)$ [see (V-4a)] as a reaction torque

$$(\vec{t}_o)_r = \overline{\langle \delta\vec{M} \times \delta\vec{H} \rangle} \tag{VI-1}$$

or, in terms of the Fourier components of $\delta\vec{M}$ and $\delta\vec{H}$ [see (V-6a)],

$$(\vec{t}_o)_r = \sum_{\vec{k}\neq 0} \overline{\vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}}} \quad . \tag{VI-1-1}$$

We recall that the bar denotes an ensemble average and the bracket a spatial average. The spatial components of $\vec{m}_{\vec{k}}$ are

$$\vec{m}_{\vec{k}} = M_o(\vec{i}_\Theta \Theta_{\vec{k}} + \vec{i}_\varphi \varphi_{\vec{k}}) \quad , \tag{VI-2}$$

while the spatial components of $\vec{h}_{-\vec{k}}$ are found through (V-7) as linear functions of $\Theta_{-\vec{k}}$ and $\varphi_{-\vec{k}}$. Using (V-59), we see that $\vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}}$ will contain terms of the following four types:

$$|\Theta_{\vec{k}}|^2 \quad ; \quad |\varphi_{\vec{k}}|^2 \quad ; \quad \Theta_{\vec{k}}\varphi^*_{\vec{k}} \quad ; \quad \varphi_{\vec{k}}\Theta^*_{\vec{k}} \quad . \tag{VI-3}$$

For τ in $\mathfrak{R}_\infty$ (to which we shall confine ourselves in this section), we have seen that the spin-wave components $\varphi_{\vec{k}}(\tau)$ and $\Theta_{\vec{k}}(\tau)$ oscillate at a frequency $\omega_{\vec{k}}(\tau)$ much greater than the precessional frequency of $\vec{m}_o$. Therefore, $\vec{m}_o$ will respond, not to the instantaneous reaction torque, but to an effective torque which is averaged over the period of this oscillation. Since $f(\tau)$ is slowly varying, we may consider it to be constant during one period of oscillation. Then since $\Theta_{\vec{k}}$ and $\varphi_{\vec{k}}$ are 90° out of phase, terms of the last two types in (VI-3) give no contribution to the effective torque, and we are left with

$$\vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}} = M_o\left[\vec{i}_m \mathcal{H}^{\Theta\varphi}(|\Theta_{\vec{k}}|^2 - |\varphi_{\vec{k}}|^2) + \vec{i}_\Theta \mathcal{H}^{m\varphi}\,|\varphi_{\vec{k}}|^2 - \vec{i}_\varphi \mathcal{H}^{m\Theta}\,|\Theta_{\vec{k}}|^2\right] \tag{VI-4}$$

where the factors $\sin^2(g - \pi/6)$ and $\cos^2(g - \pi/6)$ of $|\Theta_{\vec{k}}|^2$ and $|\varphi_{\vec{k}}|^2$, respectively, are to be replaced by their average value $\frac{1}{2}$. We note that exchange fields (which appear in the diagonal elements of $\overleftrightarrow{\mathcal{H}}_{\vec{k}}$) do not contribute to the effective reaction torque. Also, since $H_K << 4\pi M_o \tilde{\chi}$ [except for extremely long wavelength components which, we will find, make no significant contribution $(\vec{t}_o)_r$], the anisotropy term in $\overleftrightarrow{\mathcal{H}}_{\vec{k}}$ may be safely neglected, leaving only magneto-static terms. Equations (VI-4), (V-22), (V-23a, b), and (V-26) then give

$$\begin{aligned}\vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}} = 4\pi M_o^2 \{ &\vec{i}_m \psi_o \tilde{\chi}\ \tfrac{1}{2} \sin 2(\varphi_o - \Phi)\ (|\varphi_{\vec{k}}|^2 - |\Theta_{\vec{k}}|^2) \\ &+ \vec{i}_\Theta \tilde{\chi}\ \tfrac{1}{2} \sin 2(\varphi_o - \Phi)\ |\varphi_{\vec{k}}|^2 - \vec{i}_\varphi \psi_o\ [1 - \tilde{\chi} \sin^2(\varphi_o - \Phi)]\ |\Theta_{\vec{k}}|^2 \} \quad .\end{aligned} \tag{VI-5}$$

We now show that only the $\vec{i}_\Theta$-component of $(\vec{t}_o)_r$ can have a physically important effect on $\vec{m}_o$. Taking the ensemble average of $\vec{m}_{\vec{k}} \times \vec{h}_{-\vec{k}}$ and performing the sum over $\vec{k}$, we write

$$(\vec{t}_o)_r = 4\pi M_o^2 (\vec{i}_m \psi_o s_m + \vec{i}_\Theta s_\Theta + \vec{i}_\varphi \psi_o s_\varphi) \tag{VI-6}$$

where†

$$s_m(\varphi_o) = \frac{1}{2} \sum_{\vec{k}\neq 0} \tilde{\chi}(kL) \sin 2(\varphi_o - \Phi) \left(\overline{|\varphi_{\vec{k}}|^2} - \overline{|\Theta_{\vec{k}}|^2}\right) \tag{VI-7a}$$

† Strictly speaking, the ensemble averages in (VI-7) should include the factors containing φ_o. However, as was noted on p. 21, random fluctuations of φ_o vanish in the limit $S \to \infty$ and are therefore of no physical importance.

$$s_\Theta(\varphi_o) = \frac{1}{2} \sum_{\vec{k}\neq 0} \tilde{\chi}(kL) \sin 2(\varphi_o - \Phi) \overline{|\varphi_{\vec{k}}|^2} \tag{VI-7b}$$

$$s_\varphi(\varphi_o) = - \sum_{\vec{k}\neq 0} [1 - \tilde{\chi}(kL) \sin^2(\varphi_o - \Phi)] \overline{|\Theta_{\vec{k}}|^2} \quad . \tag{VI-7c}$$

From (I-1a), (V-5a), and (VI-2) we obtain the condition

$$1 >> \frac{\overline{\langle|\delta\vec{M}|^2\rangle}}{M_o^2} = \sum_{\vec{k}\neq 0} \left(\overline{|\Theta_{\vec{k}}|^2} + \overline{|\varphi_{\vec{k}}|^2}\right) \tag{VI-8}$$

from which we deduce the separate conditions

$$\sum_{\vec{k}\neq 0} w_{\vec{k}} \overline{|\Theta_{\vec{k}}|^2} << 1 \tag{VI-9a}$$

$$\sum_{\vec{k}\neq 0} w_{\vec{k}} \overline{|\varphi_{\vec{k}}|^2} << 1 \tag{VI-9b}$$

where $w_{\vec{k}}$ is any weighting function satisfying

$$|w_{\vec{k}}| \leqslant 1 \quad . \tag{VI-9c}$$

[Note that with $w_{\vec{k}} = 1$, (VI-9b) leads to the restriction (III-58) on the magnetization dispersion.] Since $0 < \tilde{\chi} \leqslant 1$, these conditions may be applied to the sums of (VI-7), and we see that

$$|s_j(\varphi_o)| << 1 \qquad j = m,\ \Theta,\ \varphi \quad . \tag{VI-10}$$

Next we insert the effective reaction torque of (VI-6) into the equations of motion (V-4a) for $\vec{m}_o$ and, with the help of (V-43a-c) and (V-48a, b) obtain

$$\frac{\dot{m}_o}{m_o} = -\sigma_o s_m(\varphi_o) \tag{VI-11a}$$

$$\dot{\sigma}_o = \rho(\varphi_o) - \epsilon^{-2} s_\Theta(\varphi_o) \tag{VI-11b}$$

$$\dot{\varphi}_o = \sigma_o [1 - s_\varphi(\varphi_o)] \tag{VI-11c}$$

where we have neglected random fluctuations of m_o and σ_o, as well as φ_o. (See footnote, p. 62.) We note first, from (VI-11a), that $|\vec{m}_o| = m_o(\tau)$ is no longer a constant of the motion ($|\vec{M}| = M_o$ still is, but now M_o^2 and m_o^2 differ by $\overline{\langle|\delta\vec{M}|^2\rangle}$). However, the relative change in m_o during switching is small (since $|s_m| << 1$), and no profound effects are anticipated. We next note, from (VI-11c), that the direct effect of spin waves on φ_o is small (and therefore of no physical importance); with the help of (VI-11c) we may integrate (VI-11a), obtaining the implicit solution

$$m_o(\tau) = m_o(0) \exp - \int_{\varphi_o(0)}^{\varphi_o(\tau)} s_m(\xi)\, d\xi \quad . \tag{VI-12}$$

However, because the change in m_o is probably not measurable, we shall carry this solution no further (although evaluation of s_m is straightforward) and turn instead to the one potentially large spin-wave effect — that of s_Θ.

Since $\epsilon^{-2} >> 1$ ($\sim 10^4$), the second term on the right side of (VI-11b), which comes from the $\vec{i}_\Theta$-component of the reaction torque, is not a priori negligible and, in fact, could even dominate the uniform field torque term $\rho(\varphi_o)$. To compute s_Θ we take $\varphi_{\vec{k}}(\tau)$ as given by (V-85) and, recalling the definition of λ (V-53), obtain

$$s_\Theta(\varphi_o) = \frac{1}{2} (\epsilon^2 \rho_o)^{1/3} C_\infty^2 \sum_{\vec{k} \neq 0} \tilde{\chi}^{2/3} \chi^{-1/3} \cos(\varphi_o - \Phi) \overline{|\varphi_{\vec{k}}(0)|^2} \quad . \tag{VI-13}$$

The initial ripple amplitudes are given by (III-39) and, converting the sum to an integral [see (III-44)], we find

$$s_\Theta(\varphi_o) = (\epsilon^2 \rho_o)^{1/3} \left(\frac{\Gamma r_o C_\infty}{4\Lambda}\right)^2 \frac{1}{\pi} \int_0^\infty \tilde{\chi}^{2/3} \chi^{-1/3} (1 + r_o^2 k^2)^{-3/2}$$

$$\times\, k dk \int_{\varphi_o(0)-\pi/2}^{\varphi_o(0)+\pi/2} (1 + g_{\vec{k}})^{-2} \cos(\varphi_o - \Phi)\, d\Phi \quad . \tag{VI-14}$$

We have used (V-59) to relate $\varphi_{-\vec{k}}$ to $\varphi_{\vec{k}}$, and it must be remembered that the solution for $\varphi_{\vec{k}}(\tau)$ we are integrating is valid only for $|\Psi| << \Psi_o$. The second integral in (VI-14) is

$$I_\Psi = \int_{-\pi/2}^{\pi/2} (1 + \lambda_e^2 k^2 + \lambda_m L^{-1} \tilde{\chi} \sin^2 \Psi)^{-2} \cos[\eta(\tau) - \Psi]\, d\Psi \simeq \frac{\pi}{2} \frac{\cos \eta(\tau)}{(a^3 b)^{1/2}} \tag{VI-15}$$

where

$$a = 1 + \lambda_e^2 k^2 \tag{VI-16a}$$

$$b = \lambda_m L^{-1} \tilde{\chi} \tag{VI-16b}$$

and we have made use of the fact that for $k = O(\lambda_e^{-1})$, $b >> a$. The main contribution to I_Ψ is from $\Psi^2 b/a < 1$; using (III-37), (V-74-1), and (V-53), we see that for components with $k = O(\lambda_e^{-1})$

$$\Psi_o^2 \frac{b}{a} \simeq \frac{h_p}{8\Lambda\chi} (\lambda \rho_o)^{2/3} \quad . \tag{VI-17}$$

Since χ, Λ, and h_p are all ~ 1, if $\lambda >> \rho_o^{-1}$ (we recall that $\lambda >> 1$ and the limit $\rho_o \to 0$ is excluded), then

$$\Psi_o^2 \frac{b}{a} >> 1 \tag{VI-18}$$

and only components for which our solution is valid contribute to I_Ψ. But for typical values of the parameters involved, (VI-18) is satisfied only marginally, e.g., $\Psi_o^2 b/a \simeq 2$. However, as

noted in the footnote on p. 58, the solution may be extended to $|\Psi| \leq \Psi_o$ without serious error. We estimate the possible error in I_Ψ to be +20 percent.

Equation (VI-14) now becomes

$$s_\Theta = \frac{1}{2} (\epsilon^2 \rho_o)^{1/3} \left(\frac{\Gamma r_o C_\infty}{4\Lambda}\right)^2 \left(\frac{L}{\lambda_m}\right)^{1/2} I_k \cos \eta(\tau) \tag{VI-19a}$$

where

$$I_k = \int_o^\infty \tilde{\chi}^{1/6} \chi^{-1/3} [(1 + r_o^2 k^2)(1 + \lambda_e^2 k^2)]^{-3/2} \, kdk \quad . \tag{VI-19b}$$

We evaluate I_k by the same method used to find δ^2 on pp. 27-29, with the result

$$I_k \simeq L^{1/6} \int_o^\infty (1 + \lambda_e^2 k^2)^{-3/2} k^{7/6} \, dk$$

$$= L^{1/6} \lambda_e^{-13/6} I(\tfrac{7}{6}, \tfrac{3}{2}) \tag{VI-20}$$

where I(p, q) is given by (III-65). Thus

$$s_\Theta = (\epsilon^2 \rho_o)^{1/3} \frac{C_\infty^2 \Gamma(\frac{13}{12}) \Gamma(\frac{5}{12})}{2\sqrt{\pi}} \left(\frac{\Gamma r_o}{4\Lambda}\right)^2 \lambda_m^{-1/2} \lambda_e^{-13/6} L^{2/3} \cos \eta(\tau) \quad . \tag{VI-21}$$

It is convenient to express s_Θ in terms of the zero field magnetization dispersion δ_o, given by (III-66) or (III-67) with $\Lambda = 1$, and we finally obtain for the reaction torque contribution to $\dot{\sigma}_o$

$$\epsilon^{-2} s_\Theta = h_p^{-2/3} d_o R \cos \eta(\tau) \tag{VI-22a}$$

where

$$d_o = \rho_o^{1/3} \Lambda^{-5/12} \tag{VI-22b}$$

$$R = C_r \left(\frac{M_o^4 L^2}{K_o A}\right)^{1/3} \delta_o^2 \tag{VI-22c}$$

and

$$C_r = (\tfrac{9}{\pi})^{1/3} \Gamma(\tfrac{13}{12}) \Gamma(\tfrac{5}{12}) \left[\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{3}{4})}\right]^2 = 2.45 \quad . \tag{VI-22d}$$

Before we investigate the effect of this reaction torque on the uniform mode, it is helpful to recall the meaning of the factors in (VI-22a). h_p is the (constant) magnitude of the pulse field $\vec{H}_p$, normalized to the uniform anisotropy field $H_K = 2K_o/M_o$. The factor d_o is a function of the initial state of the film and of the direction of $\vec{H}_p$, and with ρ_o and Λ given explicitly by (V-48e) and (III-32) is

$$d_o = \{\sin [\beta_p - \varphi_o(0)]\}^{1/3} \{\cos 2\varphi_o(0) + h_b \cos [\beta_b - \varphi_o(0)]\}^{-5/12} \quad . \tag{VI-23}$$

For the case of a hard-axis bias field and an easy-axis pulse field [see (IV-19a, c)]

$$d_o = h_\perp^{1/3}(1 - h_\perp^2)^{-5/12} \tag{VI-23-1}$$

and for the case of zero bias [see (IV-19b)]

$$d_o = (\sin\beta)^{1/3} \quad . \tag{VI-23-2}$$

The reaction torque coefficient R is a function of the thickness and magnetic parameters of the film; for a TPF, $R = 45\delta_o^2 = 0.26$. The last factor, $\cos\eta(\tau) = \cos[\varphi_o(\tau) - \varphi_o(0)]$, introduces an implicit time dependence into the reaction torque. The change in sign of the torque at $\eta = \pi/2$ is expected, since for $\vec{k} \parallel \vec{m}_o(0)$ the magnetostatic field and planar spin-wave components are parallel, producing no reaction torque.

C. ROTATIONAL MAGNETIZATION REVERSAL: DYNAMIC LOCKING

The equations of motion of the uniform mode, as modified by the ripple, are [from (VI-11b, c) and (VI-22a)]

$$\boxed{\begin{aligned}\dot{\sigma}_o(\tau) &= \rho[\eta(\tau)] - V\cos\eta(\tau) \qquad &\text{(VI-24a)}\\ \dot{\eta}(\tau) &= \sigma_o(\tau) \quad , &\text{(VI-24b)}\end{aligned}}$$

where the uniform torque [from (V-48b)] is

$$\rho(\eta) = -\sin[\eta + \varphi_o(0) - \beta_p] - \frac{h_b}{h_p}\sin[\eta + \varphi_o(0) - \beta_b] - \frac{1}{2h_p}\sin 2[\eta + \varphi_o(0)] \tag{VI-24c}$$

and

$$V = h_p^{-2/3} d_o R \tag{VI-24d}$$

providing τ is in $\mathcal{R}_\infty$, given by (V-83a-c). (The initial reaction torque is not V, but zero, since $\vec{M}$ is in equilibrium.) If V is small ("small" will be defined below), rather than solve (VI-24a, b) we compare them to the corresponding equations with phenomenological damping, (IV-27a, b), and verify the observation made in Sec. VI-A, namely that the reaction torque results in an effective damping in the first quadrant and an effective anti-damping in the second. An explicit solution for $\eta(\tau)$ would then show retardation in the first quadrant and acceleration in the second, with a net effect of somewhat slower switching since, without damping, the greater portion of the time for reversal is spent in the first 90°. This may account for the anomalous damping observed in coherent rotation (see p. 2); however, to compare theory and experiment on this point the intrinsic damping torque $(-\nu\sigma_o)$ should be included on the right side of (VI-24a), and a machine computation becomes necessary.

But now suppose that

$$V\cos\eta(\tau) > \rho[\eta(\tau)] \quad \text{for} \quad 0 \leqslant \tau \leqslant \tau_\infty \tag{VI-25}$$

where τ_∞ is the smallest τ in $\mathcal{R}_\infty$ (usually $\tau_\infty \ll 1$). Then the effective torque

$$\rho_{eff}(\eta) \equiv \rho(\eta) - V\cos\eta \tag{VI-26}$$

becomes negative after a short initial buildup time for the reaction torque, and rotational reversal is prevented; the uniform mode has become locked. However, the inequality (VI-25) is sufficient but not necessary for locking, which may occur later in the reversal process. We define a critical value of V, V_c, as the minimum value for which the uniform mode becomes locked; this will occur at an angle η_c, with $0 < \eta_c < \pi/2$; V_c and η_c may be found most easily by a generalization of (IV-37):

$$\rho_{eff}(\eta_c) = 0 \tag{VI-27a}$$

$$\rho'_{eff}(\eta_c) = 0 \quad . \tag{VI-27b}$$

These relations determine η_c and either V_c (at fixed field) or, alternatively, the critical pulse field h_{pc} for irreversible rotation through η_c (at fixed R). We shall adopt the second viewpoint, since the field is conveniently varied experimentally. With the help of (VI-24c) and (VI-26), (VI-27a, b) become

$$X \sin\varphi_{oc} + Y \cos\varphi_{oc} - \frac{1}{2} \sin 2\varphi_{oc} = 0 \tag{VI-28a}$$

$$X \cos\varphi_{oc} - Y \sin\varphi_{oc} - \cos 2\varphi_{oc} = 0 \tag{VI-28b}$$

where

$$\varphi_{oc} = \eta_c + \varphi_o(0) \tag{VI-29a}$$

$$X = -h_{pc} \cos\beta_p - h_b \cos\beta_b - h_{pc}^{1/3} d_o R \sin\varphi_o(0) \tag{VI-29b}$$

$$Y = h_{pc} \sin\beta_p + h_b \sin\beta_b - h_{pc}^{1/3} d_o R \cos\varphi_o(0) \quad . \tag{VI-29c}$$

We find from (VI-28a, b) the relations

$$X = \cos^3\varphi_{oc} \tag{VI-30a}$$

$$Y = \sin^3\varphi_{oc} \tag{VI-30b}$$

provided $\varphi_{oc} > \varphi_o(\tau_\infty)$, from which we obtain the equation for the critical field

$$\boxed{X^{2/3} + Y^{2/3} = 1} \quad . \tag{VI-31}$$

In Appendix D we examine (VI-31) for two field configurations of experimental interest [(IV-19a, b)].

For $\varphi_{oc} < \varphi_o(\tau_\infty)$, locking occurs at the start of reversal – at some time between 0 and τ_∞. Then to find the critical field we would compute the reaction torque from spin waves given, not asymptotically, but by (V-75) and (V-82a, b). We shall not attempt this formidable integration, but instead make the rather crude assumption that R = 0 for $\tau < \tau_\infty$. Then in the limit $\tau_\infty \to 0$, the critical field is determined by the disappearance of the initial torque

$$\rho_{eff}(0) = 0 \tag{VI-32}$$

from which we obtain

$$h_{pc}^{2/3} = d_o R \csc [\beta_p - \varphi_o(0)] \quad . \tag{VI-33}$$

Note that (VI-33) implies a cube-law dependence of h_{pc} on the dispersion δ_o, which should be very striking if it actually exists.

We conclude with a general discussion of the critical field and its implications. It should first be noted that for $R = 0$, h_{pc} becomes h_{pt} — the usual threshold field for irreversible uniform rotation [see (V-41)], and (VI-31) reduces to the Stoner-Wohlfarth asteroid[13]

$$(-h_{pt} \cos\beta_p - h_b \cos\beta_b)^{2/3} + (h_{pt} \sin\beta_p + h_b \sin\beta_b)^{2/3} = 1 \quad . \tag{VI-34}$$

Then from (VI-29b, c), (VI-30a, b), and (VI-31)

$$\left.\frac{\partial h_{pc}}{\partial R}\right|_{R=0} = h_{pt}^{1/3} d_o \frac{\cos[\varphi_{ot} - \varphi_o(0)]}{\sin(\beta_p - \varphi_{ot})} > 0 \quad . \tag{VI-35}$$

For small R, therefore, $h_{pc} - h_{pt}$ is proportional to R, and from (VI-22c)

$$h_{pc} - h_{pt} \propto \delta_o^2 \quad \text{for} \quad R \ll 1 \quad . \tag{VI-36}$$

We next note that for

$$h_p \geqslant h_{pc} \tag{VI-37}$$

rotational switching is modified but not locked by the ripple. Thus we identify the field region (VI-37) with that of high-speed switching, or "coherent rotation" (region 3 of Fig. 1), and the critical field h_{pc} with the threshold field for coherent rotation (h_{p3} of Fig. 1). Preliminary experiments performed by T. D. Rossing and G. P. Weiss at the Laboratory have verified the dependence of this threshold on the bias field h_b, as given by (VI-31). (See Fig. D-2.) If δ_o can be measured independently, the dependence of h_{pc} on the magnetization dispersion, as given by (VI-36) or its equivalent for R not small (see Appendix D), should provide a sensitive experimental test of the theory. Analogous with the replacement suggested on p. 42 to approximately account for anisotropy effects, we may replace H, in the solutions found in Sec. IV for φ_o and σ_o, by $H_p - H_{pc}$ and thereby approximately account for both anisotropy and spin-wave effects.

For

$$h_{pt} < h_p < h_{pc} \tag{VI-38}$$

$\vec{m}_o$ starts to rotate but at some point in the first quadrant becomes dynamically locked by the magnetostatic field of spin waves propagating in the direction of $\vec{m}_o(0)$ (LMR). We identify the field region (VI-38) with that of intermediate-speed switching, or "noncoherent rotation," (region 2 of Fig. 1), and offer the following conjecture on the reversal process following locking: We have seen that the occurrence of dynamic locking depended on the relaxation time of initial ripple components not being short compared to the switching time. However, with the uniform mode locked these components can now relax, either to the lattice or to components propagating in the instantaneous direction of $\vec{m}_o$, thereby losing their large magnetostatic energy and unlocking the uniform mode. Thus we envisage a highly damped rotational process, with switching

speed controlled by the spin-wave relaxation rate, and with a threshold field h_{pt} (h_{p2} of Fig. 1). This is in agreement with experimental observations (see Sec. I), but no quantitative assessment of switching speeds expected can be made until spin-wave relaxation processes are better understood, or at least until relaxation times have been measured.

APPENDIX A
NOTATION; PARAMETERS OF A TPF

I. NOTATION

The following subscripts appear on $\vec{H}$ (magnetic field), $\vec{T}$ (torque = $\vec{M} \times \vec{H}$), and their Fourier components $h_{\vec{k}}$ and $t_{\vec{k}}$:

Subscript	Meaning
eff	effective
m	magnetostatic (omitted in Sec. 11)
e	exchange
a	anisotropy
h	external field (at an angle β to $\vec{i}_x$)
p	external pulse field (at an angle β_p to $\vec{i}_x$)

The following subscripts appear on H (external field magnitude), β (angle of external field to $\vec{i}_x$), and H (=H/H_K, external field normalized to uniform uniaxial anisotropy field $H_K = 2K_o/M_o$):

Subscript	Meaning
p	pulse
b	bias
$\|\|$	pulse along $-\vec{i}_x$ (easy axis): $\beta_{\|\|} = \pi$
$\perp$	bias along $\vec{i}_y$ (hard axis): $\beta_{\perp} = \pi/2$
t	ideal single-domain threshold
c	critical value for dynamic locking: coherent threshold

II. PARAMETERS OF A TYPICAL PERMALLOY FILM (TPF)

Parameter	Symbol	Value
Gyromagnetic ratio	γ	1.87×10^7 (oe sec)$^{-1}$
Saturation magnetization	M_o	7.96×10^2 gauss
Exchange constant	A	10^{-6} erg cm^{-1}
Uniform uniaxial anisotropy	K_o	1.6×10^3 erg cm^{-3}
Root-mean-square local anisotropy	K_1	5×10^4 erg cm^{-3}
Mean crystallite boundary spacing	r_o	1.25×10^{-6} cm
Half-thickness	L	5×10^{-6} cm

APPENDIX B
COUPLING OF RIPPLE COMPONENTS

Here we carry the iterative procedure that was used to find the equilibrium state of the magnetization one step further. We shall find that the correction thus introduced, which arises from coupling of the ripple components $\varphi_{\vec{k}}$ $(\vec{k} \neq 0)$ by random anisotropy forces (but not from contributions to the torque nonlinear in $\varphi_{\vec{k}}$, which are still assumed negligible), is small provided fluctuations of the magnetization from its mean direction are small or, more precisely, provided

$$\delta^2 \equiv \sum_{\vec{k}\neq 0} \overline{|\varphi_{\vec{k}}|^2} << 1 \quad . \tag{B-1}$$

In order to evaluate the mean square amplitudes $\overline{|\varphi_{\vec{k}}|^2}$, we will require ensemble averages of the form $\overline{f_{\vec{k}} f_{\vec{k}'-\vec{k}}}$ where $f_{\vec{k}}$ is the Fourier transform of a random function $F(\vec{r})$ satisfying (III-9a-c). To find this we first obtain a generalization of the autocorrelation function [see (III-10)]

$$c_{\vec{k}}(\vec{r}) = \frac{1}{S} \int \overline{F(\vec{r}' + \vec{r})\, F(\vec{r}')}\, e^{-i\vec{k}\cdot r'}\, d^2\vec{r}' \quad . \tag{B-2}$$

Following the same argument used to derive $c_o(\vec{r})$, we find

$$\begin{aligned} c_{\vec{k}}(\vec{r}) &= \overline{F^2}\, e^{-r/r_o} \frac{1}{S} \int e^{-i\vec{k}\cdot\vec{r}'}\, d^2\vec{r}' \\ &= \overline{F^2}\, e^{-r/r_o}\, \delta_{\vec{k},0} \\ &= c_o(\vec{r})\, \delta_{\vec{k},0} \end{aligned} \tag{B-3}$$

and using the inverse Fourier transform

$$f_{\vec{k}} = \frac{1}{S} \int F(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^2\vec{r} \tag{B-4}$$

we have the result

$$\begin{aligned} \overline{f_{\vec{k}} f_{\vec{k}'-\vec{k}}} &= \frac{1}{S^2} \iint \overline{F(\vec{r})\, F(\vec{r}')}\, e^{-i[\vec{k}\cdot\vec{r} + (\vec{k}'-\vec{k})\cdot\vec{r}']}\, d^2\vec{r}\, d^2\vec{r}' \\ &= \frac{1}{S} \int c_{\vec{k}'}(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^2\vec{r} \\ &= \overline{|f_{\vec{k}}|^2}\, \delta_{\vec{k}',0} \quad . \end{aligned} \tag{B-5}$$

We now turn to (III-23b) for the ripple components $\varphi_{\vec{k}}$, which we write in the form

$$\varphi_{\vec{k}}^{(n+1)} = \frac{p_{\vec{k}}}{2U_{\vec{k}}} - \frac{1}{U_{\vec{k}}} \sum_{\vec{k}' \neq 0} q_{\vec{k}-\vec{k}'} \varphi_{\vec{k}'}^{(n)} \qquad (\vec{k} \neq 0) \tag{B-6}$$

where

$$U_{\vec{k}} = U_{-\vec{k}} = K_o \cos 2\varphi_o + M_o H \cos(\beta - \varphi_o) + Ak^2 + 2\pi M_o^2 \tilde{\chi} \sin^2(\Phi - \varphi_o)$$

$$= K_o \Lambda (1 + g_{\vec{k}}) \tag{B-7}$$

and $\varphi_{\vec{k}}^{(n)}$ is the n^{th} approximation to $\varphi_{\vec{k}}$. Starting with

$$\varphi_{\vec{k}}^{(0)} = 0 \tag{B-8}$$

we rederive (III-35):

$$\varphi_{\vec{k}}^{(1)} = \frac{p_{\vec{k}}}{2U_{\vec{k}}} \quad . \tag{B-9}$$

Then the next iteration† gives

$$\varphi_{\vec{k}}^{(2)} = \frac{1}{2U_{\vec{k}}} \left(p_{\vec{k}} - \sum_{\vec{k}' \neq 0} \frac{q_{\vec{k}-\vec{k}'} p_{\vec{k}'}}{U_{\vec{k}'}} \right) \tag{B-10}$$

with mean-square amplitude

$$\overline{|\varphi_{\vec{k}}^{(2)}|^2} = \overline{|\varphi_{\vec{k}}^{(1)}|^2} + \frac{1}{4U_{\vec{k}}^2} \left[\left(\sum_{\vec{k}' \neq 0} \frac{\overline{q_{\vec{k}-\vec{k}'} p_{\vec{k}'} p_{-\vec{k}}}}{U_{\vec{k}'}} + c.c. \right) \right.$$

$$\left. + \sum_{\vec{k}' \neq 0} \sum_{\vec{k}'' \neq 0} \frac{\overline{q_{\vec{k}-\vec{k}'} q_{\vec{k}''-\vec{k}} p_{\vec{k}'} p_{-\vec{k}''}}}{U_{\vec{k}'} U_{\vec{k}''}} \right] \quad . \tag{B-11}$$

To find the ensemble averages in (B-11) we first observe, with the help of (B-4), that $\overline{q_{\vec{k}-\vec{k}'} p_{\vec{k}'} p_{-\vec{k}}}$ may be written as a sum of terms each of which has a factor

$$\overline{Q_{m_1} P_{m_2} P_{m_3}} \tag{B-12}$$

† At this level of approximation there is also a correction to φ_o, through the term $-2 \sum_{\vec{k} \neq 0} q_{-\vec{k}} \varphi_{\vec{k}}$ of (III-23a), where $\varphi_{\vec{k}}$ here is $\varphi_{\vec{k}}^{(1)}$. This leads to a correction to $\varphi_{\vec{k}}^{(2)}$ which, however, may be neglected since it is non-linear in $\varphi_{\vec{k}}^{(1)}$.

where $Q_{m_1} = Q(\vec{r})$ with $\vec{r}$ in cell m_1, etc. However, from (III-5a, b) we see that

$$\overline{P} = \overline{Q} = \overline{QP} = \overline{QP^2} = 0 \tag{B-13}$$

and therefore each factor (B-12) vanishes (whether or not any of the m_j are equal) so that

$$\overline{q_{\vec{k}-\vec{k}'} p_{\vec{k}'} p_{-\vec{k}}} = 0 \quad . \tag{B-14}$$

Similarly,

$$\overline{Q_{m_1} Q_{m_2} P_{m_3} P_{m_4}} = \overline{Q^2}\,\overline{P^2}\,\delta_{m_1, m_2}\delta_{m_3, m_4} \tag{B-15}$$

so that the last ensemble average in (B-11) may be split:

$$\begin{aligned} \overline{q_{\vec{k}-\vec{k}'} q_{\vec{k}''-\vec{k}} p_{\vec{k}'} p_{-\vec{k}''}} &= \overline{q_{\vec{k}-\vec{k}'} q_{\vec{k}''-\vec{k}}}\;\overline{p_{\vec{k}'} p_{-\vec{k}''}} \\ &= \overline{|q_{\vec{k}-\vec{k}'}|^2}\;\overline{|p_{\vec{k}'}|^2}\,\delta_{\vec{k}'',\vec{k}'} \end{aligned} \tag{B-16}$$

where we have used (B-5) for the second step in (B-16). Inserting (B-14) and (B-16) in (B-11), we have the result

$$\overline{|\varphi_{\vec{k}}^{(2)}|^2} = \overline{|\varphi_{\vec{k}}^{(1)}|^2} + \frac{1}{4U_{\vec{k}}^2} \sum_{\vec{k}'\neq 0} \frac{\overline{|q_{\vec{k}-\vec{k}'}|^2}\;\overline{|p_{\vec{k}'}|^2}}{U_{\vec{k}'}^2} \quad . \tag{B-17}$$

But now let us assume (to be verified below) that the main contribution to the sum in (B-17) is from components with

$$|\vec{k}'| \ll r_o^{-1} \quad . \tag{B-18}$$

Then we see from (III-14) that $\overline{|q_{\vec{k}-\vec{k}'}|^2}$ is independent of $\vec{k}'$, and since $\overline{P^2} = \overline{Q^2}$, (B-17) may be written

$$\overline{|\varphi_{\vec{k}}^{(2)}|^2} = \overline{|\varphi_{\vec{k}}^{(1)}|^2} + \frac{\overline{|p_{\vec{k}}|^2}}{4U_{\vec{k}}^2} \sum_{\vec{k}'\neq 0} \frac{\overline{|p_{\vec{k}'}|^2}}{U_{\vec{k}'}^2} \quad . \tag{B-19}$$

Finally, using (B-9) and (B-1), we obtain the main result of this appendix

$$\boxed{\overline{|\varphi_{\vec{k}}^{(2)}|^2} = \overline{|\varphi_{\vec{k}}^{(1)}|^2}\,(1 + 4\delta^2)} \tag{B-20}$$

where δ is the magnetization dispersion with $\varphi_{\vec{k}} = \varphi_{\vec{k}}^{(1)}$. The assumption (B-18) is then justified for exactly the same reasons that the term $(r_o k)^2$ was neglected in (III-60).

We conclude that

$$\overline{|\varphi^{(2)}_{\vec{k}}|^2} \simeq \overline{|\varphi^{(1)}_{\vec{k}}|^2} \qquad \text{(B-21)}$$

and it seems unlikely that further iterations will change $\overline{|\varphi_{\vec{k}}|^2}$ by very much. Thus, unless (B-1) is violated (in which case the entire theory breaks down since the nonlinear torque terms will no longer be small), the assumption we have made – that ripple components are uncoupled in static equilibrium (and also dynamically → see p. 43) – is indeed correct.

APPENDIX C
ON THE WKB METHOD

Consider the general linear, homogeneous, second-order differential equation, which may be put in the normal form

$$\varphi'' + F(x)\,\varphi = 0 \quad . \tag{C-1}$$

We first show that a pair of solutions to (C-1) is

$$\varphi(x) = h(x)\, J_{\pm\nu}[g(x)] \tag{C-2}$$

provided a generating function g(x) can be found for the potential F(x) such that

$$F(x) = (g')^2 - \frac{3}{4}\left(\frac{g''}{g'}\right)^2 + \frac{1}{2}\,\frac{g'''}{g'} + \left(\frac{1}{4} - \nu^2\right)\left(\frac{g'}{g}\right)^2 \tag{C-3a}$$

$$= (g')^2 - \frac{r''}{r} \tag{C-3b}$$

where

$$r = \left(\frac{g^{1-2\nu}}{g'}\right)^{1/2} \quad ; \tag{C-3c}$$

h is given by

$$h(x) = \left(\frac{g}{g'}\right)^{1/2} \quad . \tag{C-4}$$

The proof requires only the Bessel function recursion relations

$$\frac{2\nu}{g}\, J_\nu(g) = J_{\nu-1}(g) + J_{\nu+1}(g) \tag{C-5a}$$

$$2\,\frac{d}{dg}\, J_\nu(g) = J_{\nu-1}(g) - J_{\nu+1}(g) \quad ; \tag{C-5b}$$

differentiating (C-2) twice with the help of (C-5a, b), we find

$$\varphi'' = \left[h'' - h(g')^2\left(1 - \frac{\nu^2}{g^2}\right)\right] J_{\pm\nu}(g) + \left[h'g' + \frac{1}{2}\,hg'' - \frac{1}{2}\,\frac{h}{g}\,(g')^2\right]$$

$$\times\ \left[J_{\pm\nu-1}(g) - J_{\pm\nu+1}(g)\right] \quad , \tag{C-6}$$

so that (C-1) is satisfied if

$$h'' - h(g')^2\left(1 - \frac{\nu^2}{g^2}\right) = -Fh \tag{C-7a}$$

$$h'g' + \frac{1}{2}\,hg'' - \frac{1}{2}\,\frac{h}{g}\,(g')^2 = 0 \quad . \tag{C-7b}$$

Equation (C-7b) may be written

$$\frac{h'}{h} = \frac{1}{2}\left(\frac{g'}{g} - \frac{g''}{g'}\right) \tag{C-8}$$

which we integrate, obtaining (C-4) (where a multiplicative constant, clearly of no interest, has been set equal to unity). Finally, eliminating h from (C-4) and (C-7a) we arrive at (C-3a).

Equation (C-3a) has the significance that every g(x) generates an exactly soluble potential F(x), which depends on ν. If no g can be found for a given F, it may be possible with some ν to find a $\tilde{g}$ such that $\tilde{F}$ approximates F (and has the same zeros and poles) in some range $\mathfrak{R}$ of x. Then

$$\tilde{\varphi}(x) = \left(\frac{\tilde{g}}{\tilde{g}'}\right)^{1/2} J_{\pm\nu}[\tilde{g}(x)] \tag{C-9}$$

will be an approximate solution of (C-1) in $\mathfrak{R}$. We give a few examples of generating functions and their potentials:

(a) $g(x) = Cx^{\mu} \qquad (\mu \neq 0)$

$F(x) = (C\mu x^{\mu-1})^2 \qquad \text{for} \quad \nu = \frac{1}{2\mu}$

(b) $g(x) = C\, e^{\mu x}$

$F(x) = \mu^2(C^2\, e^{2\mu x} - \nu^2)$

(c) $g(x) = C \ln x$

$F(x) = (C^2 + \frac{1}{4})^{-2} \qquad \text{for} \quad \nu = \frac{1}{2}$

$\quad = (\frac{1}{4} - \nu^2)\,(x \ln x)^{-2} \qquad \text{for} \quad C = \pm\frac{i}{2}$

We also note the asymptotic behavior of the solution (C-2), which follows from the asymptotic expansion of the Bessel functions $J_{\pm\nu}(g)$. For

$$g = |g|\, e^{i\phi} \tag{C-10a}$$

$$|g| \gg \frac{1}{2}\, |\frac{1}{4} - \nu^2| \tag{C-10b}$$

we find[56]

$$\varphi(x) \to \sqrt{\frac{2}{\pi g'}}\, \cos[g - \frac{\pi}{2}(\pm\nu + \frac{1}{2})] \qquad -\pi < \phi < \pi \tag{C-11a}$$

$$\sqrt{\frac{2}{\pi g'}}\, \exp[\pi i(\pm\nu + \frac{1}{2})]\, \cos[g + \frac{\pi}{2}(\pm\nu + \frac{1}{2})] \qquad 0 < \phi < 2\pi \tag{C-11b}$$

$$\sqrt{\frac{2}{\pi g'}}\, \exp[2\pi i(\pm\nu + \frac{1}{2})]\, \cos[g - \frac{\pi}{2}(\pm\nu + \frac{1}{2})] \qquad \pi < \phi < 3\pi \quad , \tag{C-11c}$$

etc.

This approach to the solution of (C-1) is especially appropriate if the potential is of the WKB form

$$F(x) = [\lambda f(x)]^2 \tag{C-12a}$$

where

$$\lambda \gg 1 \tag{C-12b}$$

and f(x) and its derivatives are O(1) in some range $\mathfrak{R}_{\infty}$ of x away from "free" points f(x) = 0. In particular, if f(x) is slowly varying near a free point or has other unusual behavior there,

i.e., an inflection point, the standard methods may fail, and the procedure we now give may be useful. For x in $\Re_\infty$ we assume there exists a solution of the form

$$\varphi(x) = \exp\left[i\lambda \int_{x_o}^{x} p(\xi)\, d\xi\right] \tag{C-13}$$

where x_o is a constant. Letting $p(x) = p_o(x) + \lambda^{-1} p_1(x) + \ldots$, and dropping terms of $O(\lambda^{-2})$, we substitute (C-13) and (C-12a) in (C-1) to obtain the usual WKB asymptotic solutions

$$\varphi(x) \xrightarrow[x \text{ in } \Re_\infty]{} \frac{1}{\sqrt{f}} \exp\left[\pm i\lambda \int_{x_o}^{x} f(\xi)\, d\xi\right] \quad . \tag{C-14}$$

Comparison of (C-14) and (C-11a) shows that the asymptotic value of the generating function for a WKB potential (C-12a) is

$$g(x) \xrightarrow[x \text{ in } \Re_\infty]{} \lambda \int_{x_o}^{x} f(\xi)\, d\xi \equiv \tilde{g}(x) \tag{C-15}$$

which is easily verified by substitution in (C-3a), giving

$$F(x) = [\lambda f(x)]^2 \, [1 + O(\lambda^{-2})] \quad . \tag{C-16}$$

Now inserting (C-15) into (C-2) and (C-4), we find that

$$\varphi(x) \xrightarrow[x \text{ in } \Re_\infty]{} \left(\frac{\tilde{g}}{\lambda f}\right)^{1/2} J_{\pm\nu}(\tilde{g}) \equiv \tilde{\varphi}(x) \tag{C-17}$$

is an asymptotic solution of (C-1) and (C-12a) for any ν.

The solution (C-14) breaks down, of course, near a free point $f = 0$. But if a ν can be found such that $\tilde{g}(x)$ generates a close approximation to the potential $[\lambda f(x)]^2$ in a continuous range of x which includes both a free point and $\Re_\infty$, then $\tilde{\varphi}(x)$ will be a good approximation to $\varphi(x)$ in that range. For convenience we locate the free point in whose neighborhood we seek a solution at $x = 0$, and expand $f(x)$ about this point:

$$f(x) = x^{\mu} \sum_{j=1}^{\infty} f_j x^j \tag{C-18}$$

where $\mu > -1$. If $f(x)$ is analytic at the origin, then

$$\mu = 0 \tag{C-19a}$$

$$f_j = \frac{1}{j!} \left.\frac{d^j f}{dx^j}\right|_{x=0} \quad , \tag{C-19b}$$

and if there is a small residual potential at $x = 0$ (a "nearly free" point), a $j = 0$ term may be included in (C-18) instead of the $j = 1$ term. It is usually advantageous to have the lower limit x_o in (C-15) coincide with the free point, and with $x_o = 0$ the expansion of (C-15) about $x = 0$ is, from (C-18),

$$\tilde{g}(x) = \lambda x^{\mu+1} \sum_{j=1}^{\infty} \frac{f_j}{\mu + j + 1} x^j \quad . \tag{C-20}$$

Now suppose that for a continuous range $\mathcal{R}_m$ of x, which includes both the neighborhood of the free point and part of $\mathcal{R}_\infty$, $\tilde{g}(x)$ may be approximated by the j = m term of (C-20)

$$\tilde{g}(x) \simeq \frac{\lambda f_m}{\mu + m + 1} x^{\mu+m+1} \quad . \tag{C-21}$$

Then $\tilde{g}(x)$ generates a close approximation to $[\lambda f(x)]^2$ in $\mathcal{R}_m$ providing

$$\nu = \frac{1}{2(\mu + m + 1)} \tag{C-22}$$

as may be seen from example (a), p. 78. The requirement that $\mathcal{R}_m$ and $\mathcal{R}_\infty$ overlap is satisfied if there is some x_m in $\mathcal{R}_m$ for which condition (C-10b) holds:

$$2\lambda\nu \, |f_m| \, |x_m|^{1/2\nu} >> \frac{1}{2} \left|\frac{1}{4} - \nu^2\right| \quad . \tag{C-23}$$

It may be that no suitable $\mathcal{R}_m$ exists; for example, two terms of (C-20) might be required to connect the free point with $\mathcal{R}_\infty$. Nevertheless, it may be possible, using (C-3a), to find a $\approx{g}(x)$ which approaches $\tilde{g}(x)$ for x in $\mathcal{R}_\infty$ and generates a good approximation to $[\lambda f(x)]^2$ around the free point.

To summarize, $\tilde{\varphi}(x)$ as given by (C-17) is an approximate solution to (C-1) and (C-12a) in $\mathcal{R}_\infty$ and $\mathcal{R}_m$, with $\tilde{g}(x)$ given by (C-15) with $x_o = 0$ and ν by (C-22). The asymptotic formulas (C-11) may then be used to establish the connection between the coefficients of the solutions (C-14) in $\mathcal{R}_\infty$ on either side of the free point. The approximate potential generated by $\tilde{g}(x)$ differs from $[\lambda f(x)]^2$ by an amount given by the second term of (C-3b), and this may be used to improve the approximation if necessary.[57]

APPENDIX D
THE CRITICAL FIELD

In Sec. VI we obtained a critical field, $H_{pc} = H_K h_{pc}$, which we identified as the theoretical pulse threshold for coherent, high-speed switching Equations (VI-31) and (VI-29b, c), which determine h_{pc}, cannot in general be solved in closed form. In this appendix we use graphical and numerical methods to find h_{pc} for the two field configurations of greatest interest.

I. HARD-AXIS BIAS FIELD – EASY-AXIS PULSE FIELD

In this, the most frequently used arrangement,

$$\beta_p = \pi \quad , \quad \beta_b = \frac{\pi}{2} \tag{D-1a}$$

$$h_p \to h_{||} \quad , \quad h_b \to h_{\perp} \tag{D-1b}$$

$$\varphi_o(0) = \arcsin h_{\perp} \tag{D-1c}$$

and therefore from (VI-29b, c) and (VI-23-1)

$$X = h_{||c} - f_x R \tag{D-2a}$$

$$Y = h_{\perp} - f_y R \tag{D-2b}$$

where the reaction torque coefficient R is given by (VI-22c) and

$$f_x = h_{||c}^{1/3}\, h_{\perp}^{4/3}\, g_{\perp}^{-5/6} \tag{D-3a}$$

$$f_y = h_{||c}^{1/3}\, h_{\perp}^{1/3}\, g_{\perp}^{1/6} \tag{D-3b}$$

$$g_{\perp} = \sqrt{1 - h_{\perp}^2} = \cos\varphi_o(0) \quad . \tag{D-3c}$$

A necessary condition for Eq. (VI-31) to determine $h_{||c}$ is $\varphi_{oc} > \varphi_o(0)$, or

$$Y > h_{\perp}^3$$

which leads to

$$R < h_{\perp}^{2/3}\, g_{\perp}^{3/2} \tag{D-4}$$

where we have used the relation

$$\frac{f_x}{f_y} = \frac{h_{\perp}}{g_{\perp}} = \tan\varphi_o(0) \quad . \tag{D-5}$$

We shall assume that (D-4) is also sufficient for (VI-31) to be valid, which is equivalent to assuming that $\tau_{\infty} << 1$. Equation (VI-31) is most conveniently solved for R (given $h_{||c}$ and $h_{\perp}$) by the graphical method sketched in Fig. D-1, where we have made use of (D-5). In Fig. D-2 we plot $h_{||c}$ vs $h_{\perp}$ with R as a parameter.

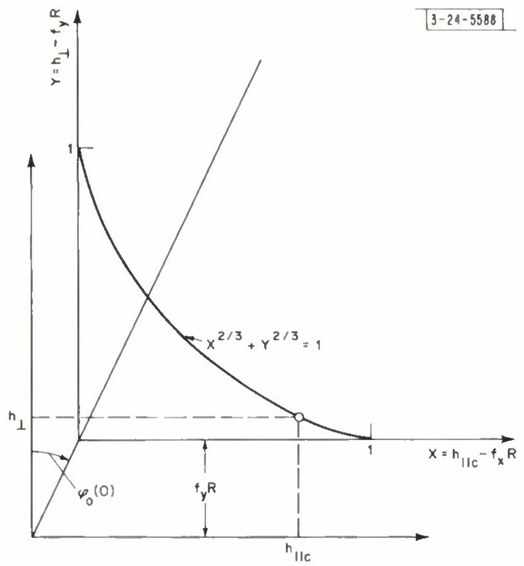

Fig. D-1. Graphical construction used to find reaction torque coefficient R as a function of hard-axis bias field $h_\perp$ and critical easy-axis switching field $h_{\|c}$.

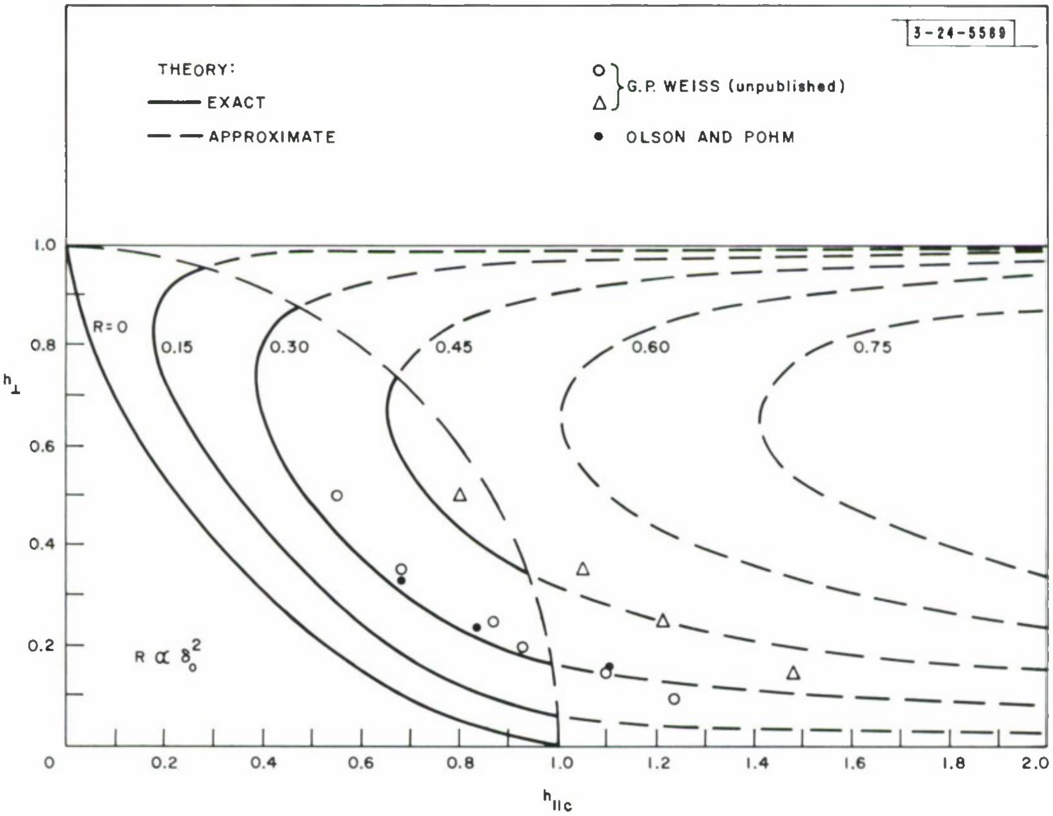

Fig. D-2. High-speed switching threshold curves with reaction torque coefficient R as a parameter: theory and experiment ($h_{p3} = h_{\|c}$ with hard-axis bias field $h_\perp$).

For

$$R > h_{\perp}^{2/3} \; g_{\perp}^{3/2} \tag{D-6}$$

$h_{||c}$ is given in first approximation by (VI-33), which becomes

$$h_{||c} = \frac{R^{3/2}}{h_{\perp} \, g_{\perp}^{5/4}} \quad . \tag{D-7}$$

However, this expression may be seriously in error since it neglects the growth time of the reaction torque; we show it as a dashed line in Fig. D-2.

II. ZERO BIAS FIELD

In this arrangement (which is more frequently used to study quasistatic than pulse switching)

$$\beta_p = \pi - \tilde{\beta} \qquad (0 < \tilde{\beta} \leq \tfrac{\pi}{2}) \tag{D-8a}$$

$$h_p \to h \quad , \qquad h_b = 0 \tag{D-8b}$$

$$\varphi_o(0) = 0$$

and therefore

$$X = h_c \cos\tilde{\beta} \tag{D-9a}$$

$$Y = h_c \sin\tilde{\beta} - (h_c \sin\tilde{\beta})^{1/3} \, R \quad . \tag{D-9b}$$

The condition for (VI-31) to determine h_c is now

$$R < (\tan\tilde{\beta})^{2/3} \quad . \tag{D-10}$$

We solve (VI-31) for $h_c \cos\tilde{\beta}$ as a function of $h_c \sin\tilde{\beta}$ and R, and the result (with R a parameter) is plotted in Fig. D-3.

For

$$R > (\tan\tilde{\beta})^{2/3} \tag{D-11}$$

(VI-33) gives the first approximation

$$h_c = \frac{R^{3/2}}{\sin\tilde{\beta}} \tag{D-12}$$

which is shown as a dashed line in Fig. D-3.

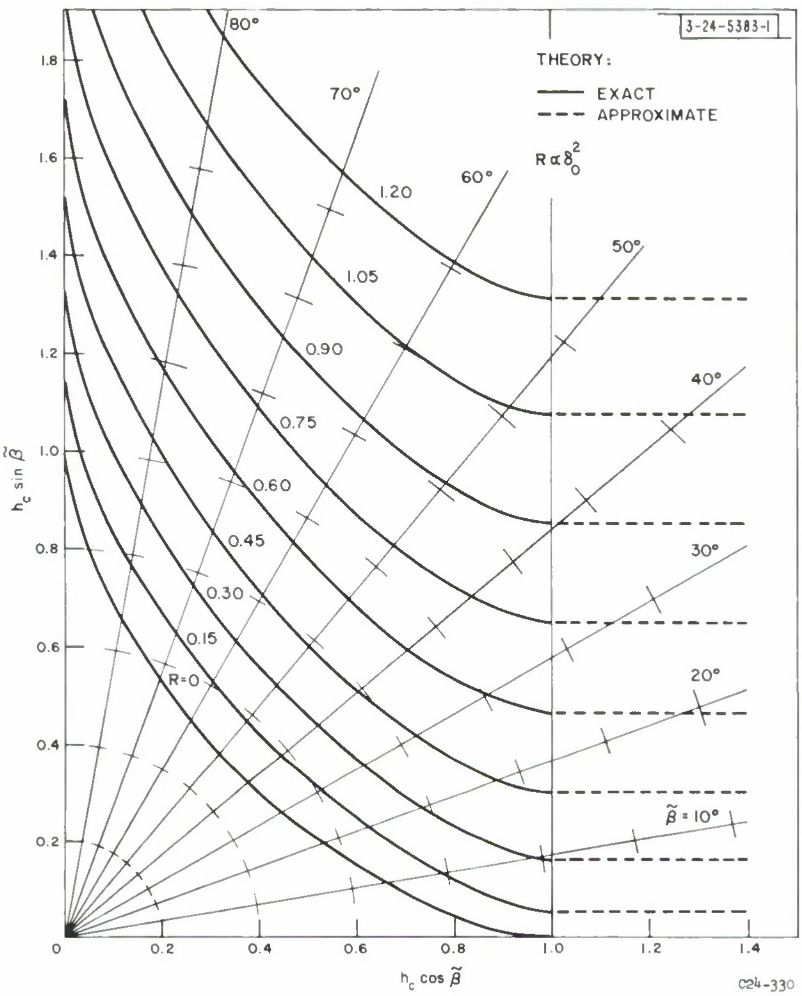

Fig. D-3. High-speed switching threshold curves with reoction torque coefficient R os porometer ($h_{p3} = h_c$ at an ongle $\tilde{\beta}$ to eosy direction opposite initial magnetizotion).

ACKNOWLEDGMENTS

It is a pleasure to acknowledge the support and advice of D.O. Smith, who originally suggested the problem around which this work developed and who provided many illuminating discussions. The author is indebted to G.F. Carrier, who provided a number of suggestions for surmounting mathematical obstacles, and to R.V. Jones for encouragement, support, and advice. Useful discussions with R.L. Orbach, T.D. Rossing, and T.A. Kaplan are gratefully acknowledged. The advice and help of Lincoln Laboratory Group 24 personnel – G. P. Weiss and M.S. Cohen, in particular – has been invaluable. Thanks are also due to Mrs. M. Gardner for computer assistance.

REFERENCES

1. M.S. Blois, Jr., J. Appl. Phys. 26, 975 (1955).
2. R.L. Conger and F.C. Essig, Phys. Rev. 104, 915 (1956).
3. D.O. Smith, Phys. Rev. 104, 1280 (1956).
4. __________, J. Appl. Phys. 29, 264 (1958).
5. C.D. Olson and A.V. Pohm, J. Appl. Phys. 29, 274 (1958).
6. F.B. Humphrey, J. Appl. Phys. 29, 284 (1958).
7. F.B. Hagedorn, J. Appl. Phys. 30, 254S (1959).
8. F.B. Humphrey and E.M. Gyorgy, J. Appl. Phys. 30, 935 (1959).
9. W. Dietrich, W.E. Proebster, and P. Wolf, IBM J. Res. Develop. 4, 189 (1960).
10. B.R. Hearn, J. Electronics Control 16, 33 (1964).
11. R.L. Conger and F.C. Essig, J. Appl. Phys. 28, 855 (1957).
12. L. Landau and E. Lifshitz, Physik. Z. Sowjetunion 8, 153 (1935).
13. E.C. Stoner and E.P. Wohlfarth, Phil. Trans. Roy. Soc. (London) A240, 599 (1948).
14. P. Wolf, J. Appl. Phys. 32, 95S (1961).
15. D.O. Smith and K.J. Harte, J. Appl. Phys. 33, 1399 (1962).
16. T.S. Crowther, J. Appl. Phys. 34, 580 (1963).
17. E.J. Torok, R.A. White, A.J. Hunt, and H.N. Oredson, J. Appl. Phys. 33, 3037 (1962).
18. T.L. Gilbert, Armour Research Foundation, Report No. 11 (25 January 1955), unpublished.
19. K.J. Harte, J. Appl. Phys. 31, 283S (1960).
20. D.O. Smith, J. Appl. Phys. 32, 70S (1961).
21. S. Methfessel, S. Middelhoek, and H. Thomas, J. Appl. Phys. 32, 1959 (1961).
22. C. Herring and C. Kittel, Phys. Rev. 81, 869 (1951).
23. C. Kittel, Phys. Rev. 76, 743 (1949); J.H. Van Vleck, Phys. Rev. 78, 266 (1950).

24. D. O. Smith, Magnetism, G. T. Rado and H. Suhl, eds. (Academic Press, New York, 1963), Vol. III, Chap. 10, pp. 506-7, 517.

25. W. F. Brown, Jr., Micromagnetics (Interscience, New York, 1963).

26. S. Middelhoek, Ferromagnetic Domains in Thin Ni-Fe Films (van Saest, Amsterdam, 1961), p. 52.

27. T. Schaug-Pettersen, J. Appl. Phys. 31, 382S (1960).

28. H. W. Fuller and M. E. Hale, J. Appl. Phys. 31, 238 (1960).

29. E. Fuchs, Z. angew. Phys. 13, 157 (1961).

30. M. S. Cohen, Fifth International Congress for Electron Microscopy, S. S. Breese, Jr., ed. (Academic Press, New York, 1962), Vol. I, p. E-9.

31. D. O. Smith, M. S. Cohen, and G. P. Weiss, J. Appl. Phys. 31, 1755 (1960).

32. R. L. Coren and H. J. Juretschke, Phys. Rev. 126, 1378 (1962).

33. H. Rother, Z. Physik 168, 42 (1962).

34. J. Koczér, M. Zelený, and P. Šudo, Czech. J. Phys. B13, 579 (1963).

35. G. W. Kenrick, Phil. Mag. 7, 176 (1929).

36. H. Thomas, J. Appl. Phys. 33, 1117 (1962).

37. See, for example, P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), p. 425.

38. F. Bloch, Z. Physik 61, 206 (1930).

39. H. Hoffmann, Phys. der kondens. Materie. 2, 32 (1964).

40. ———, phys. stat. sol. 5, 187 (1964).

41. R. Kikuchi, J. Appl. Phys. 27, 1352 (1956).

42. W. Dietrich and W. E. Praebster, J. Appl. Phys. 31, 281S (1960).

43. H. B. Callen, J. Phys. Chem. Solids 4, 256 (1958).

44. See, for example, L. M. Milne-Thomson, Jacobian Elliptic Function Tables (Dover Publications, New York, 1950).

45. See, for example, G. F. Carrier, Communs. Pure and Appl. Math. 7, 11 (1954).

46. H. Suhl, J. Phys. Chem. Solids 1, 209 (1957).

47. L. R. Walker, Phys. Rev. 105, 390 (1957).

48. R. W. Damon and J. R. Eshbach, J. Phys. Chem. Solids 19, 308 (1961).

49. K. J. Harte, Bull. Am. Phys. Soc. 7, 448 (1962).

50. M. H. Seavey, Jr. and P. E. Tannenwald, Phys. Rev. Letters 1, 168 (1958).

51. J. B. Comly, T. Penney, and R. V. Jones, J. Appl. Phys. 34, 1145 (1963).

52. ——— and T. Penney (to be published).

53. L. N. Bulaevskii, V. M. Fain, and G. I. Freidman, Zhur. Eksp. i Teor. Fiz. 39, 519 (1960) [Soviet Physics-JETP 12, 361 (1961)].

54. ———, Izv. Vysshikh Uchebn. Zavedenii. Radiofiz. 4, 639 (1961).

55. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1958) 2nd ed., p. 40.

56. Ibid, pp. 199-202.

57. Morse and Feshbach, op. cit., p. 1096.